Simulations

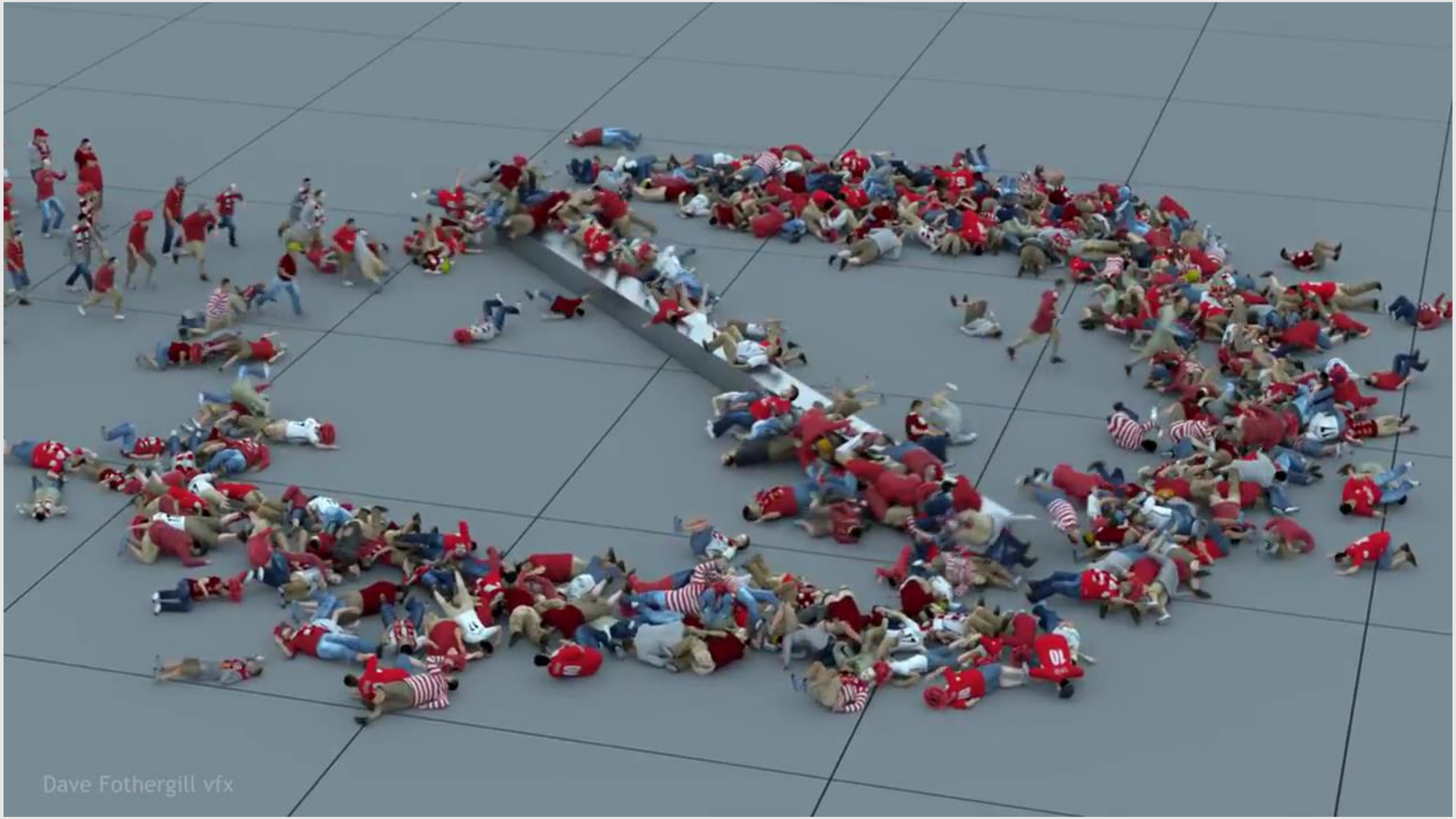
- Physically-Based Animation
- ODE Solvers

What natural phenomenon can we simulate?

Flocking Simulation

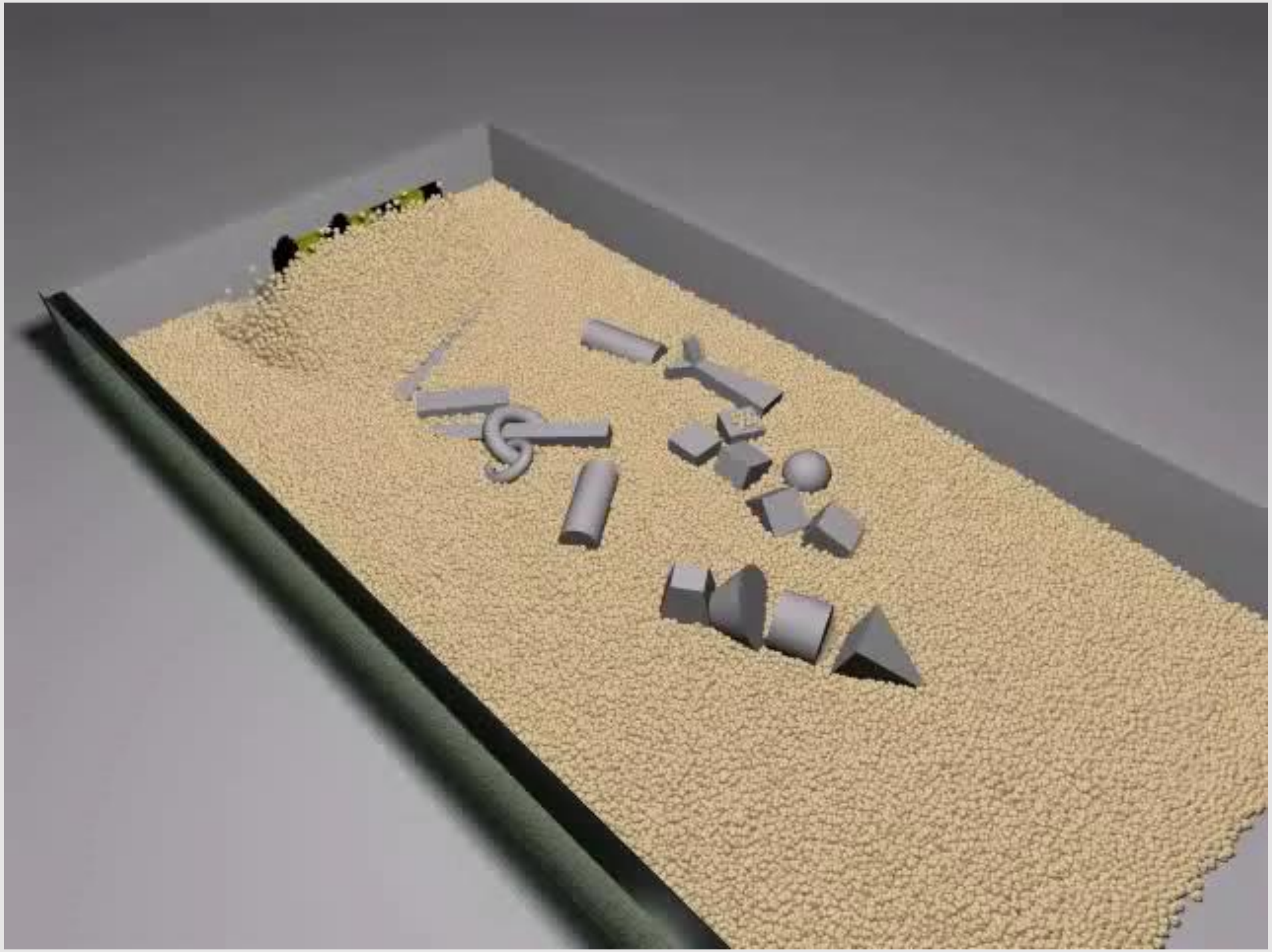
Crowd Simulation

Crowd Simulation

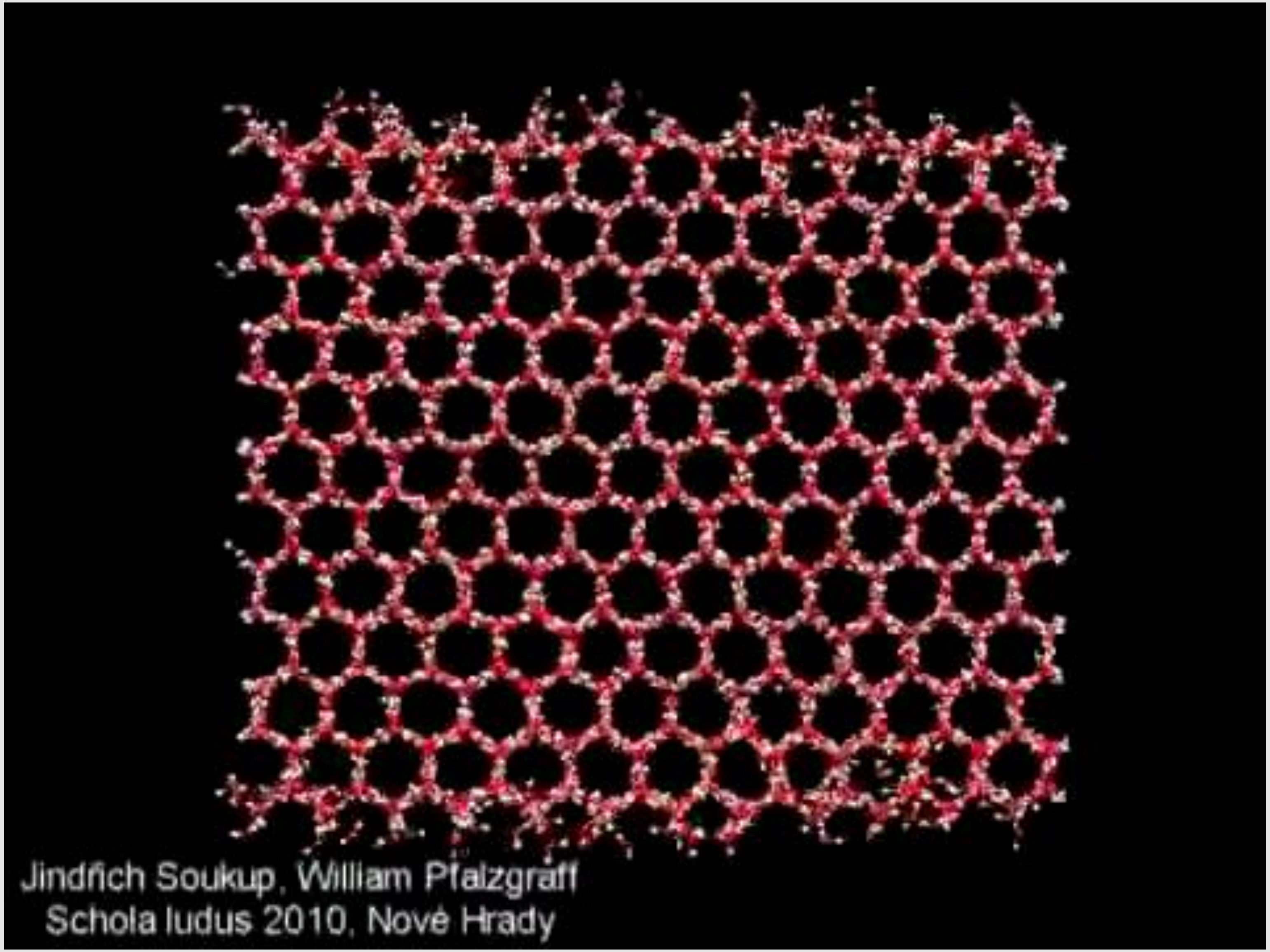


Fluid Simulation

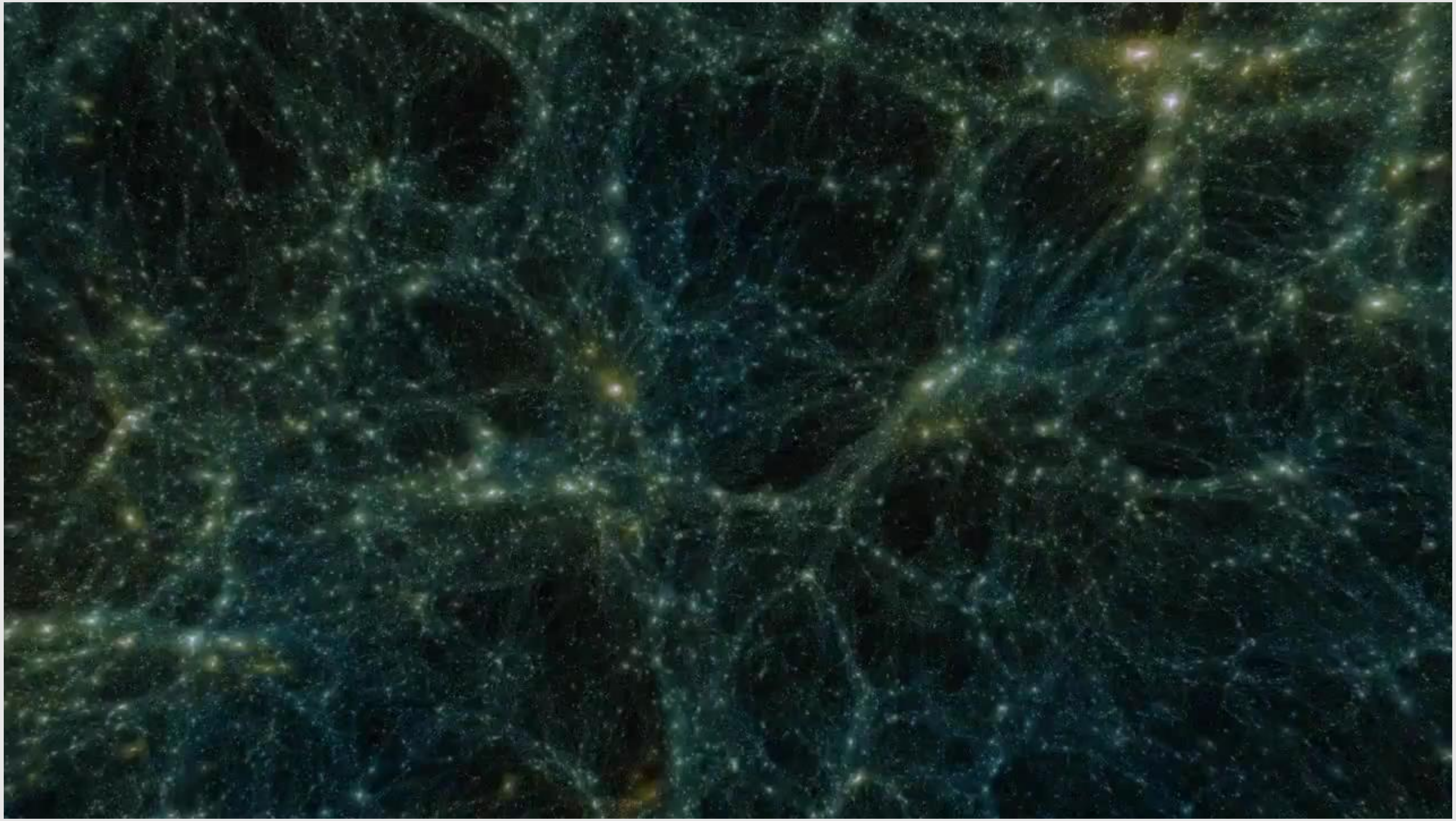
Granular Material Simulation



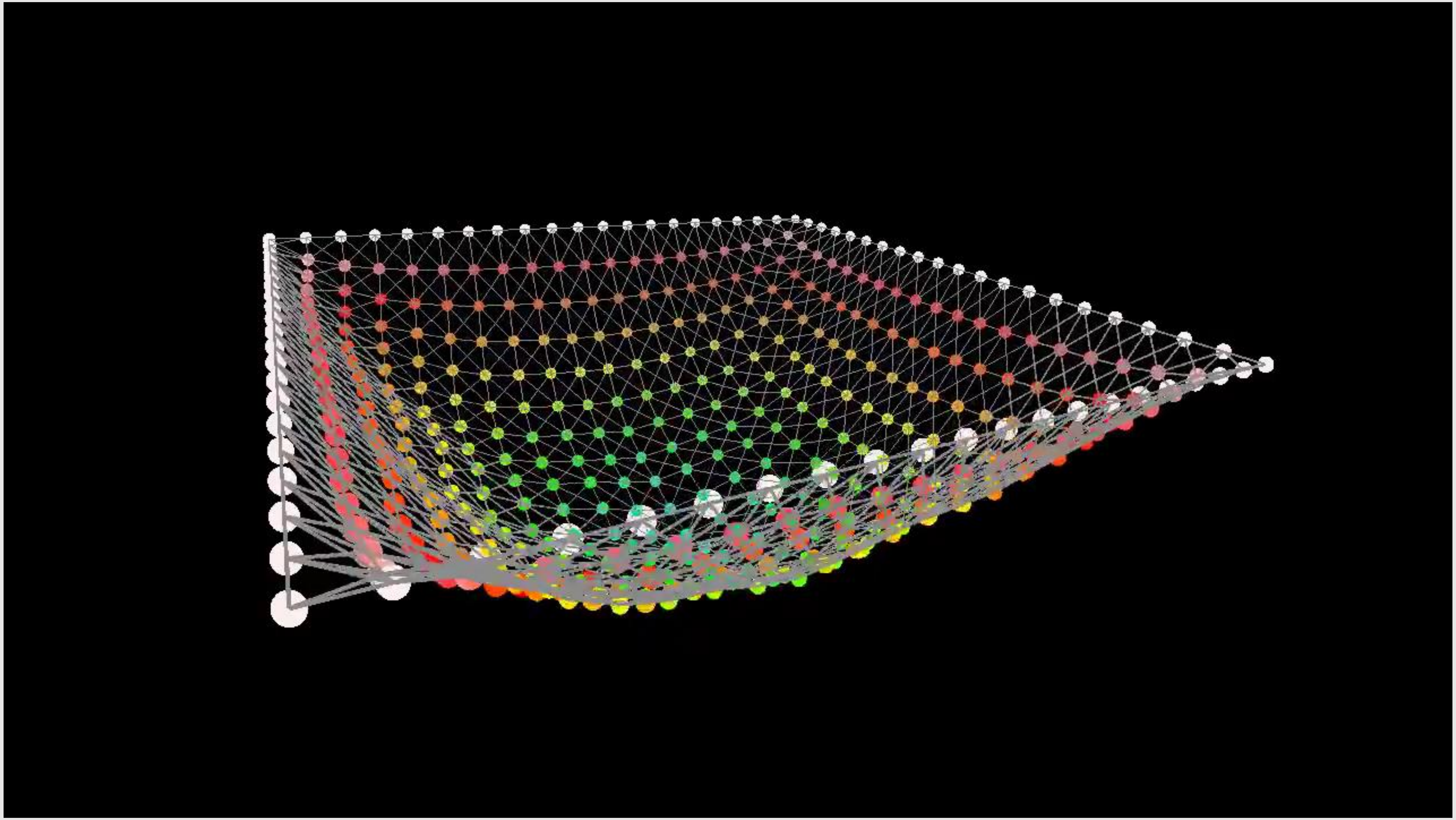
Molecular Dynamics Simulation



Cosmological Simulation

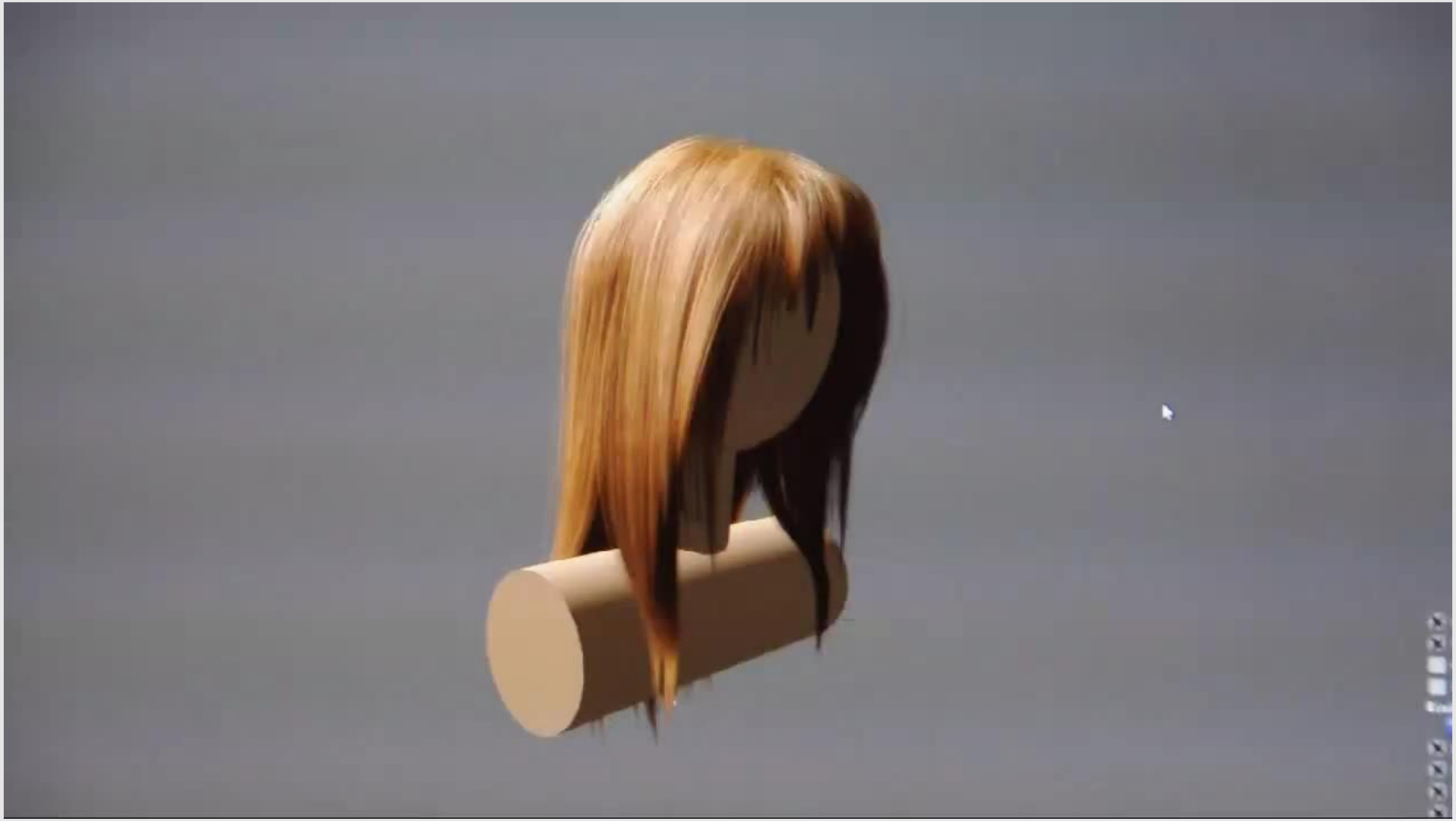


Mass-Spring Simulation

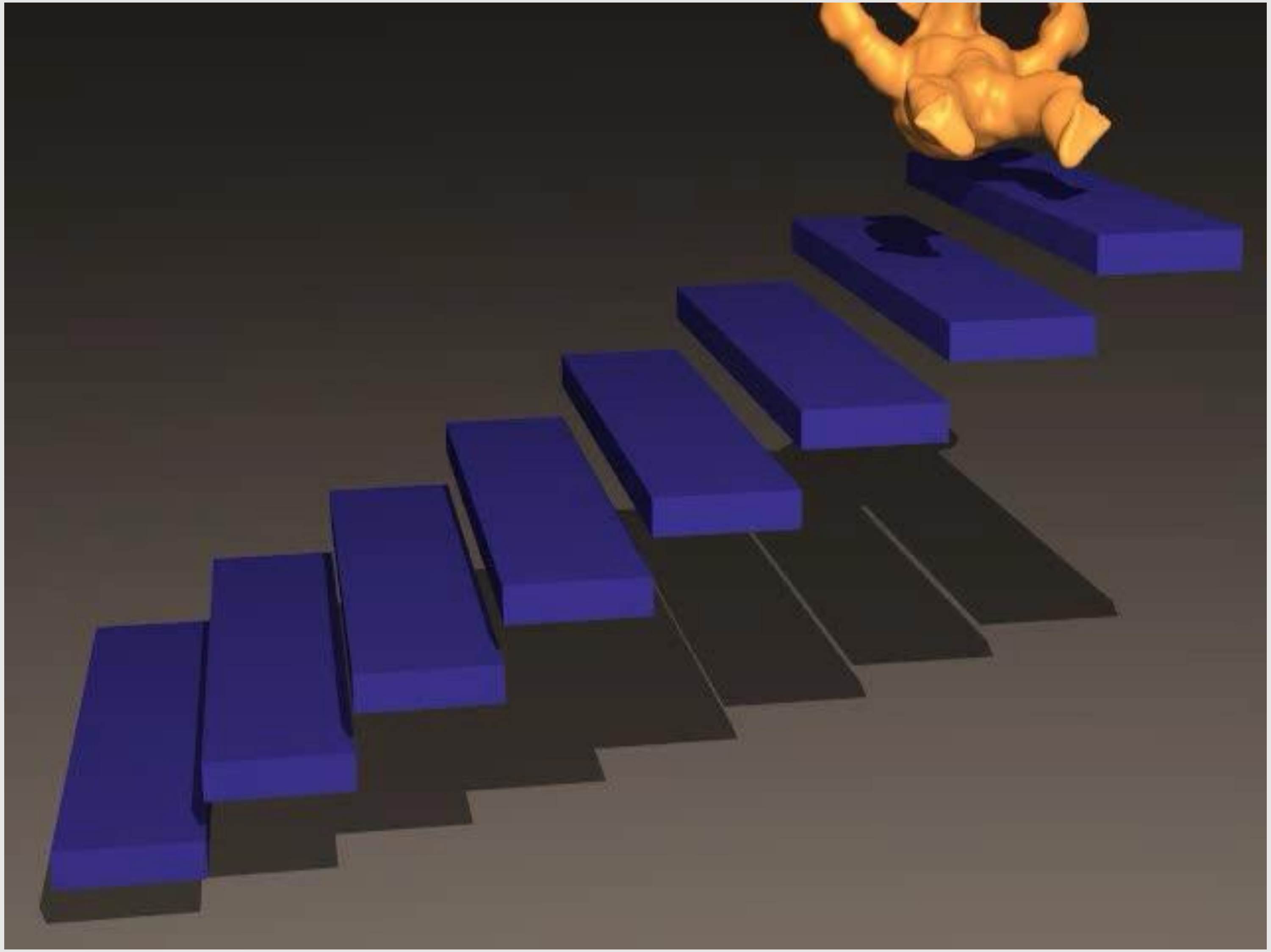


Cloth Simulation

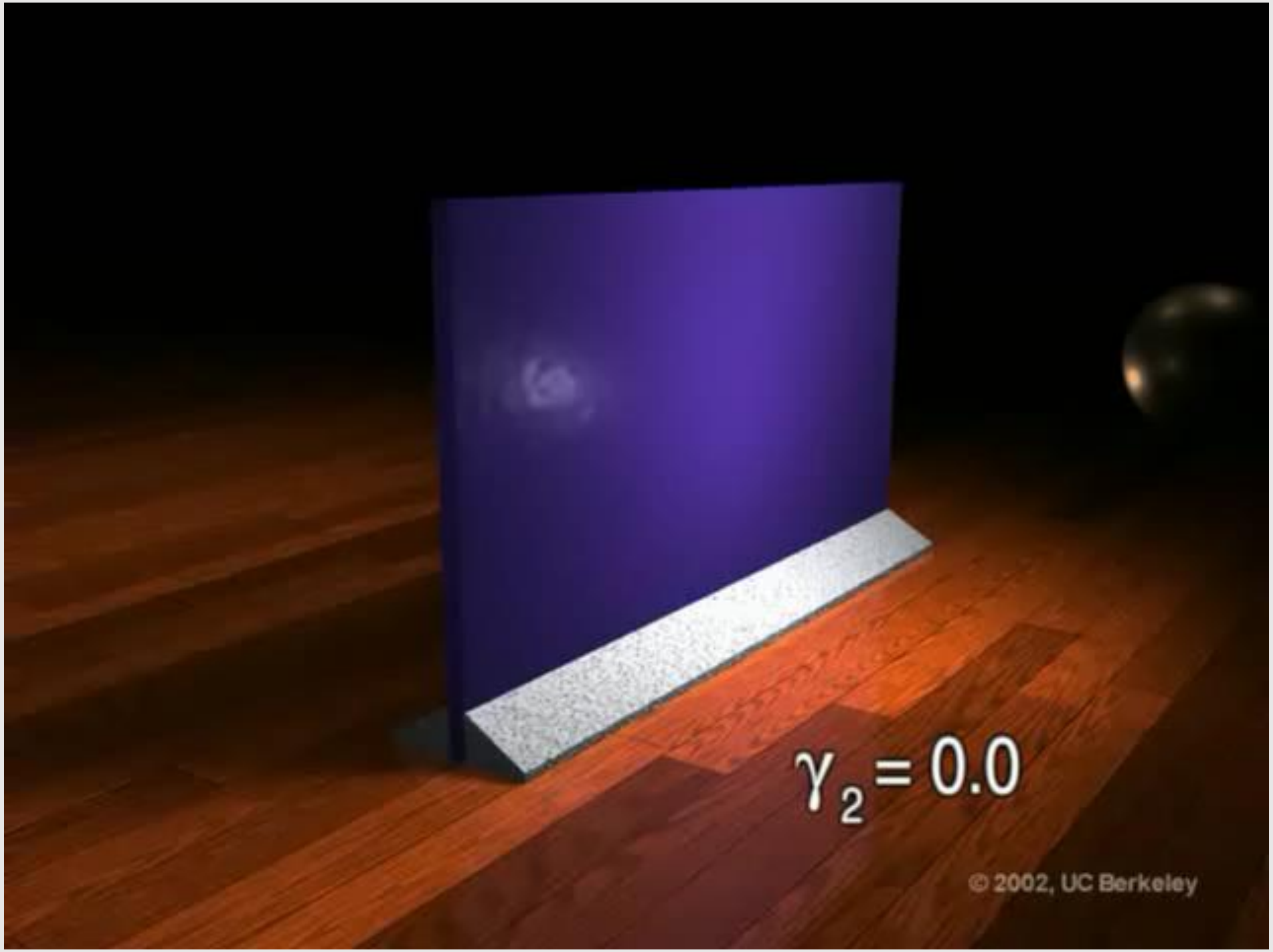
Hair Simulation



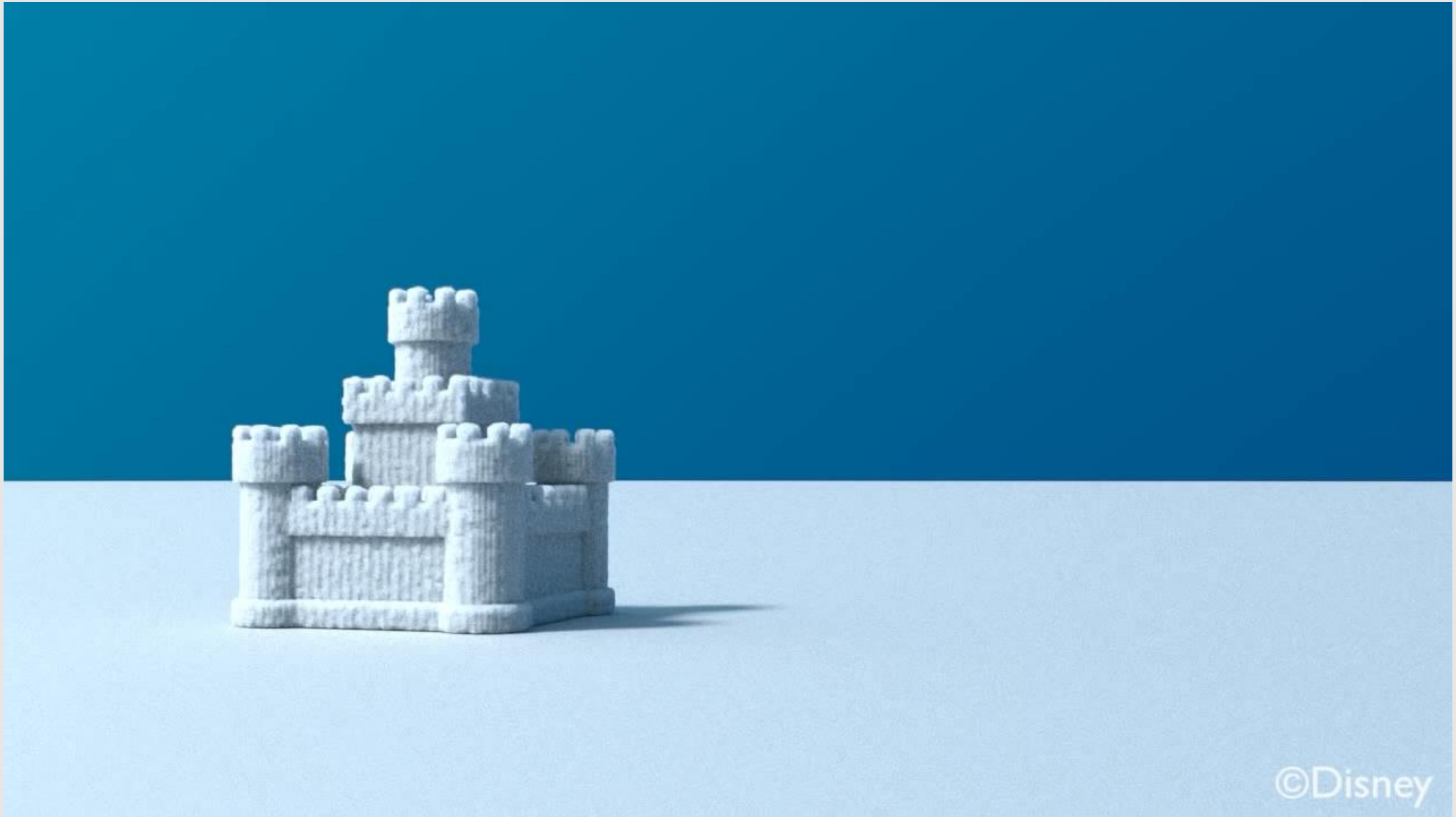
Elasticity Simulation



Fracture Simulation



Snow Simulation



Ok, simulation is cool,
How can we solve them analytically?

- ~~Physically-Based Animation~~

- ODE Solvers

Ordinary Differential Equations

- **Ordinary Differential Equations (ODEs)** have a derivative with respect to one other variable
  - Ordinary - involves derivatives in time but not space

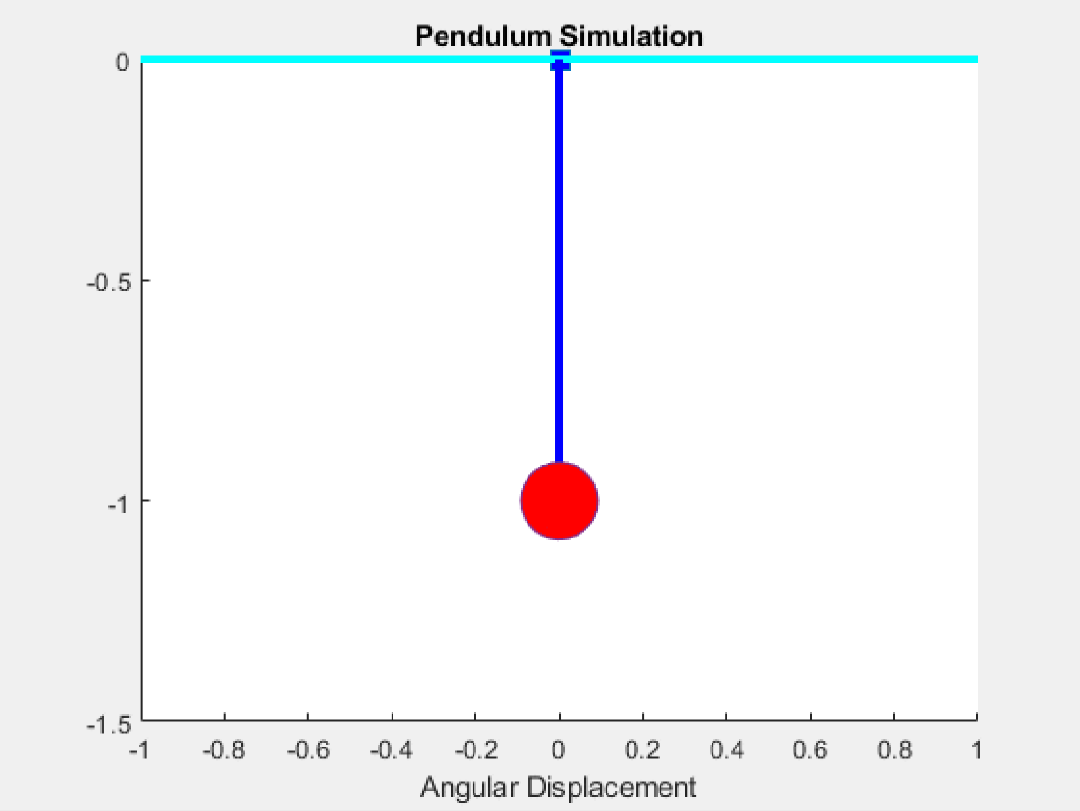
- Many dynamical systems can be described via an ODE in generalized coordinates:

$$\frac{d}{dt}q = f(q, \dot{q}, t)$$

- ODEs can also be used to model rates of growth proportional to some original value:

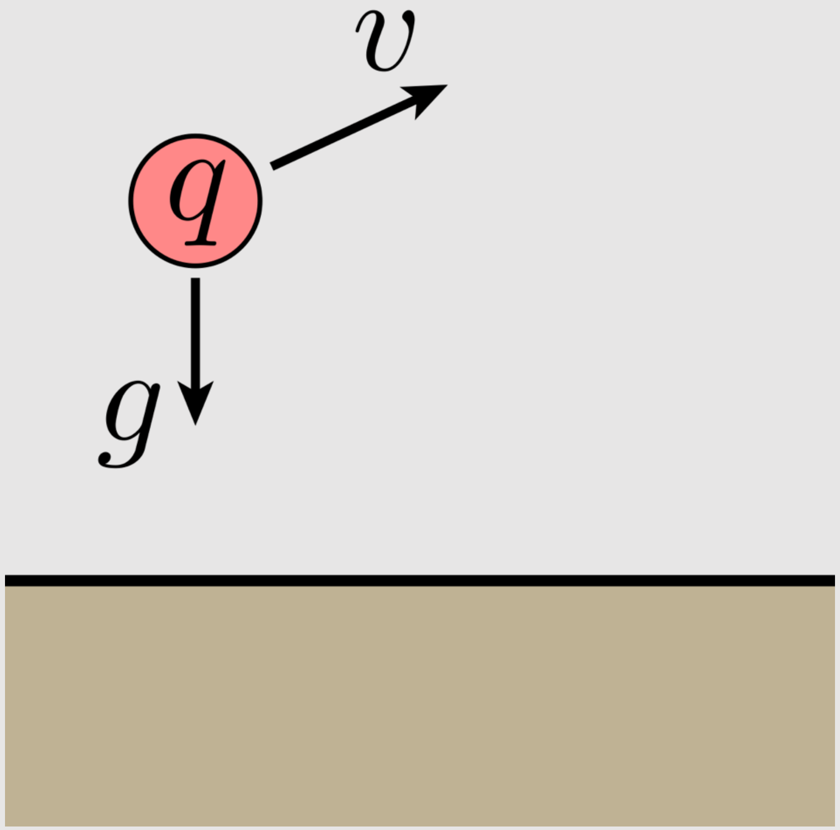
$$\frac{d}{dt}u(t) = au$$

  - **Solution:** $u(t) = be^{at}$
  - Describes exponential decay ($a < 1$), or stock ($a > 1$)



Simulation using second order ODE in MATLAB

Example: Throwing A Rock



- Consider a rock** of mass *m* tossed under force of gravity *g*
  - Easy to write dynamical equations, since only force is due to gravity:

$$\ddot{q} = g$$

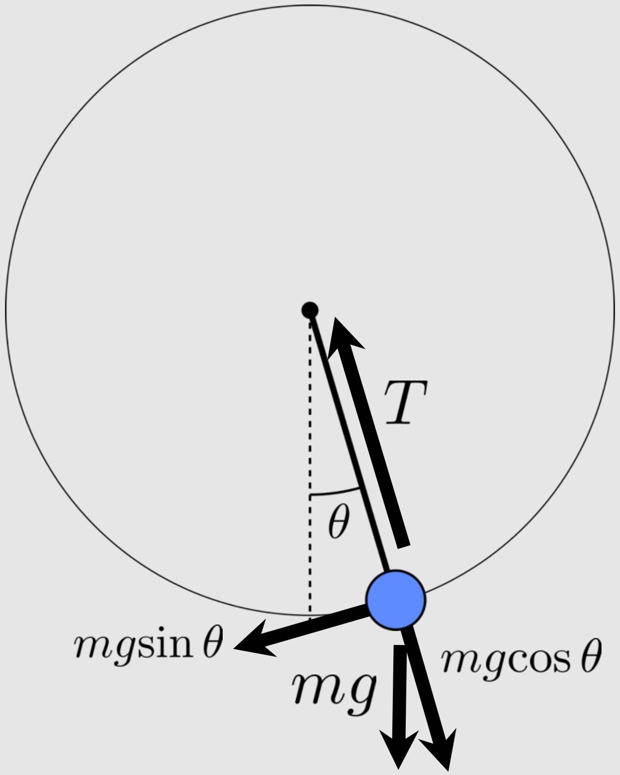
$$v(t) = v_0 + gt$$

$$q(t) = q_0 + v_0 t + \frac{gt^2}{2}$$

Easy! We don't need a computer for simulation!

** Yes, the rock is spherical and has uniform density

Example: Pendulum



- Mass on end of a bar, swinging under gravity
- What are the equations of motion?
  - Same as “rock” problem, but constrained
  - Response tension $T(q)$ now varies based on configuration q
- Could use a “force diagram”
  - You probably did this for many hours in high school/college

Ok, maybe bring back the computer…

Lagrangian Mechanics

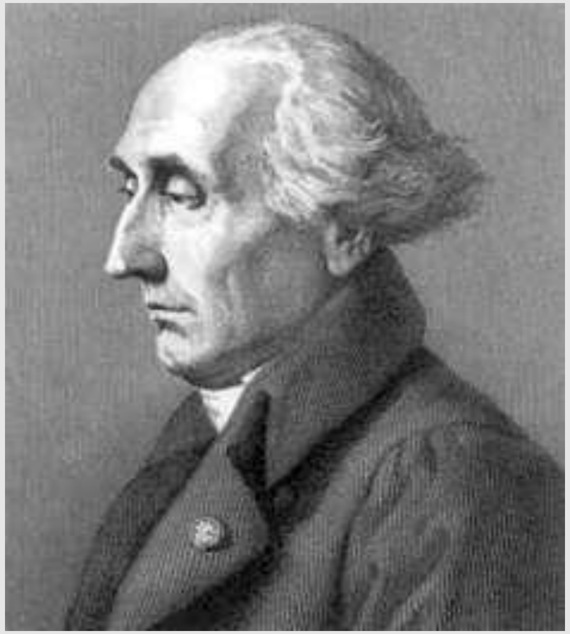
- Beautifully simple recipe:
  - Write down kinetic energy K
  - Write down potential energy U
  - Write down **Lagrangian**

$$\mathcal{L} := K - U$$

  - Dynamics then given by **Euler-Lagrange equation**

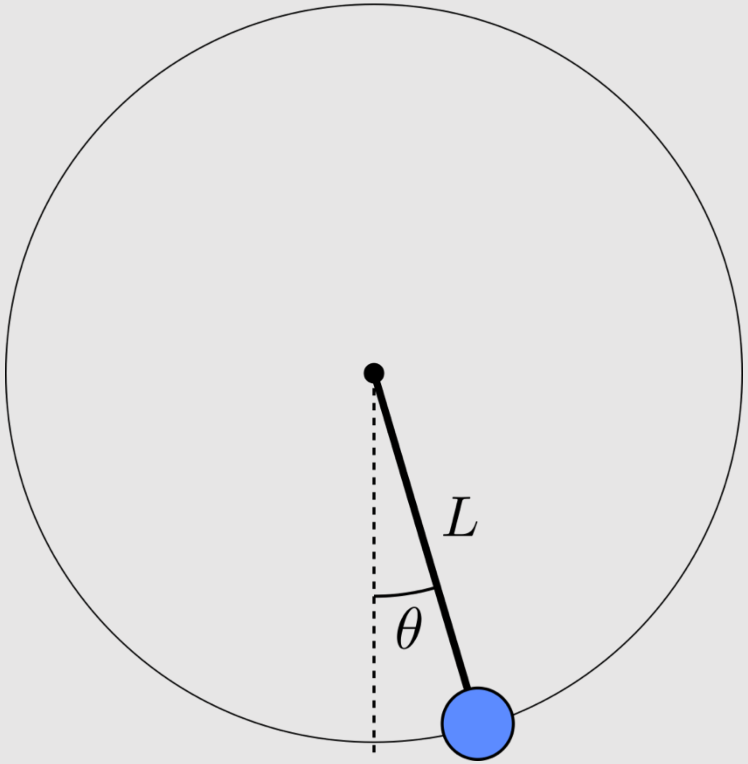
mass times acceleration → $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$ ← **force**

- Often easier to come up with (scalar) energies than forces
  - Very general, works in any kind of generalized coordinates
  - Helps develop nice class of numerical integrators (symplectic)



Joseph-Louis Langrange (1736 - 1813)

Lagrangian Mechanics: Pendulum



Simple configuration parameterization:

$$q = \theta$$

Kinetic energy:

$$K = \tfrac{1}{2} I \omega^2 = \tfrac{1}{2} m L^2 \dot{\theta}^2$$

Potential energy:

$$U = mgh = -mgL\cos\theta$$

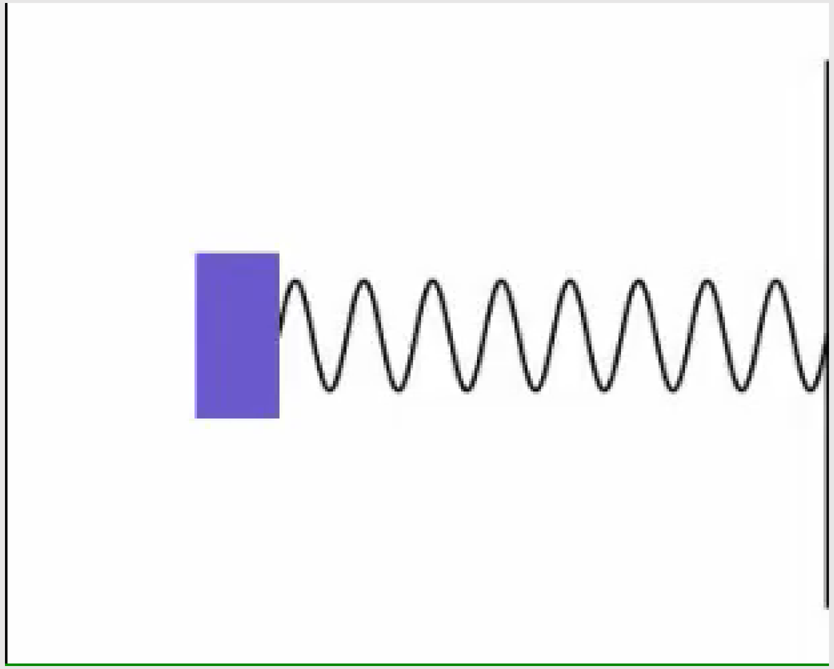
Euler-Lagrange equations:

$$\mathcal{L} = K - U = m(\tfrac{1}{2} L^2 \dot{\theta}^2 + gL\cos\theta)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\dot{\theta} \qquad \frac{\partial \mathcal{L}}{\partial q} = \frac{\partial \mathcal{L}}{\partial \theta} = -mgL\sin\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q} \quad \Rightarrow \quad \boxed{\ddot{\theta} = -\tfrac{g}{L}\sin\theta}$$

Solving The Pendulum



[harmonic oscillation]

Simple equation for the pendulum:

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

For small angles (e.g., clock pendulum) can approximate as:

$$\ddot{\theta} = -\frac{g}{L}\theta \;\Rightarrow \theta(t) = a\cos(t\sqrt{g/L} + b)$$

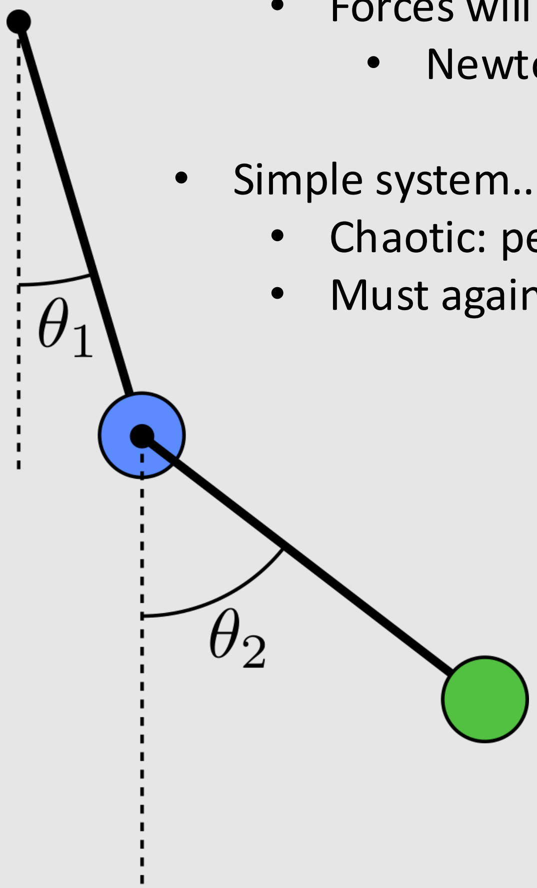
$\boldsymbol{sin\theta = \theta}$ **for small angles**

$\boldsymbol{\frac{d^2}{d\theta^2}cos\theta = -cos\theta}$

In general, there is no closed form solution!
Hence, we must use a numerical approximation

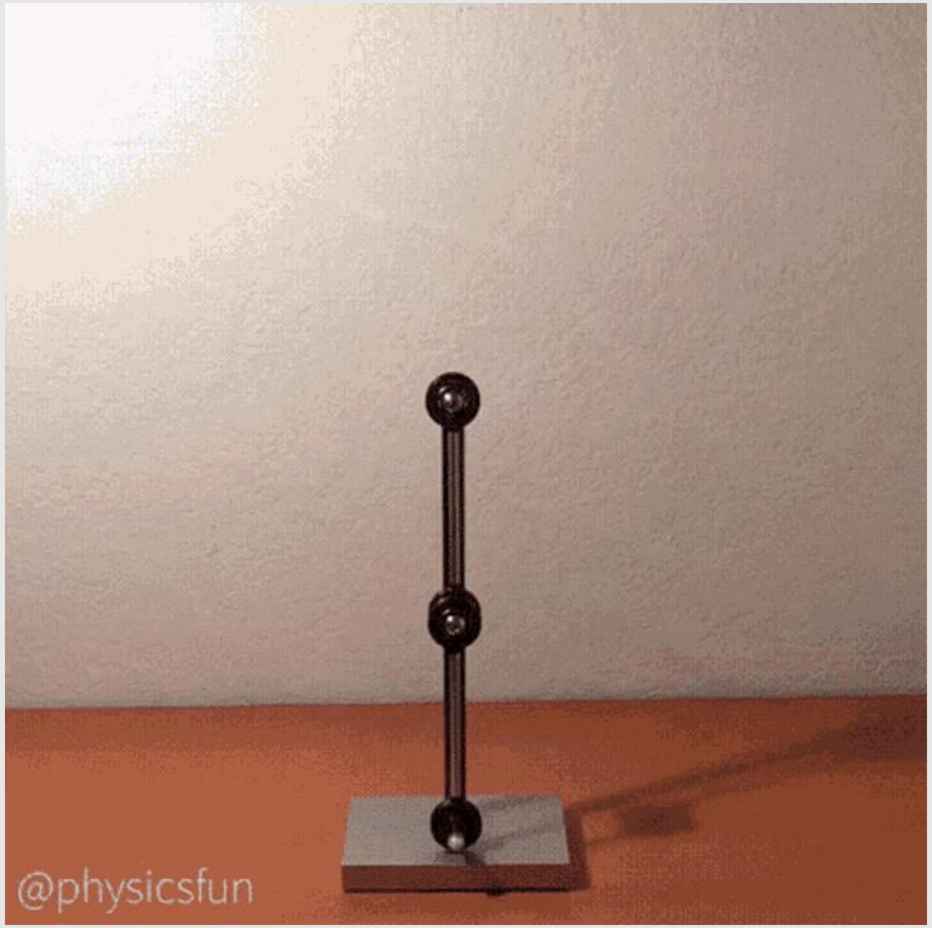
And pendulums are supposed to be easy to simulate!

Harder: Double Pendulum



- Blue ball swings from pendulum
  - Green ball swings from blue ball
  - Forces will act on each other
    - Newton's 3rd law

- Simple system...not-so-simple motion
  - Chaotic: perturb input, wild changes to output
  - Must again use numerical approximation

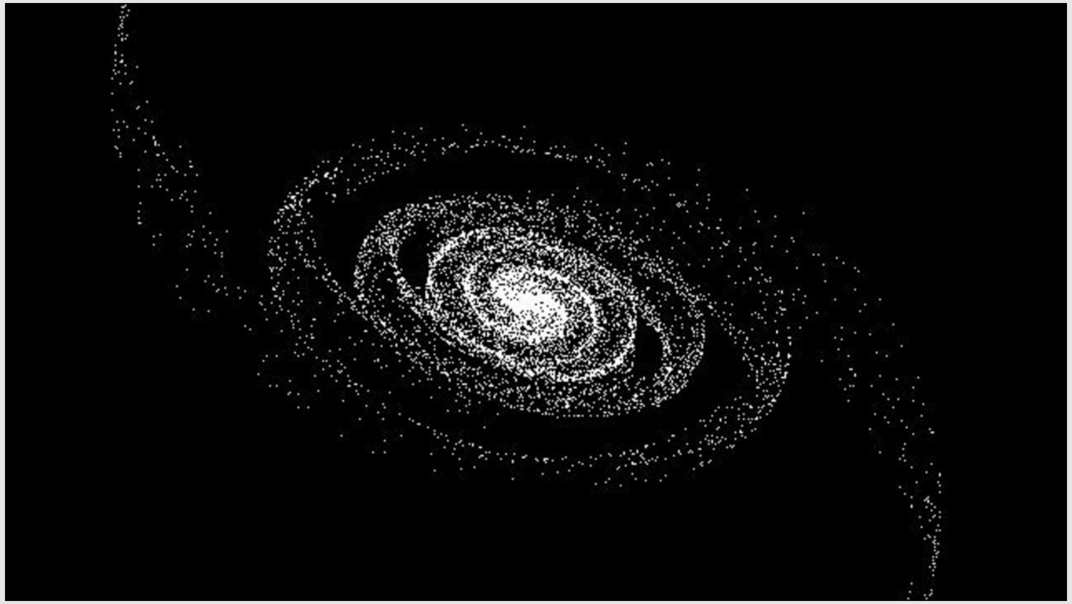


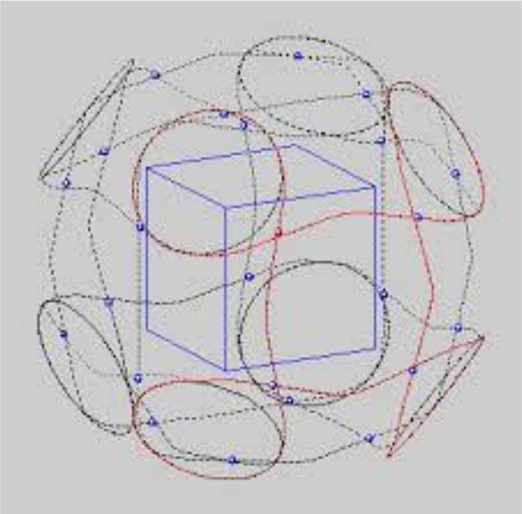
Even Harder: N-Body Problem

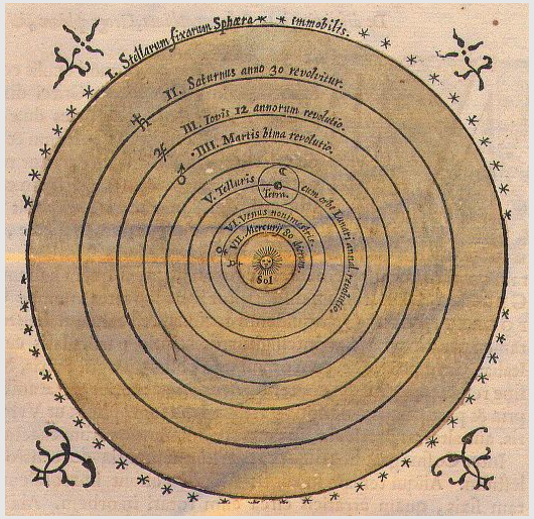
- Consider the Earth, moon, and sun
  - Where do they go?
  - Solution is trivial for two bodies
    - Assume one is fixed, solve for the other

- As soon as $n \geq 3$, gets chaotic
  - No closed form solution

- **Fun Fact:** this used to be a 15-418 homework assignment







closed-form solution

Ok, so solving solutions analytically is hard,
How can we solve them numerically?

guess-and-check

Numerical Integration

- **Key idea:** replace derivatives with differences
  - With ODEs, only need to worry about derivative in **time**
- Replace time-continuous configuration function $q(t)$ with samples q_k in time

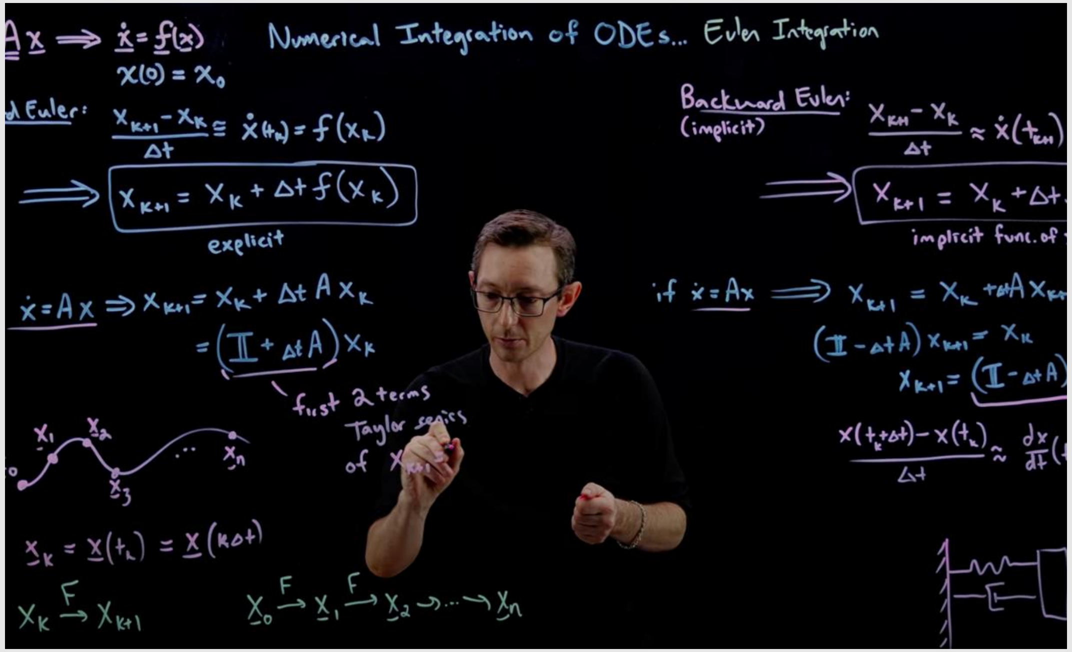
which q do we use?

new config (need to solve for)

current config

$$v(q(t)) = \frac{d}{dt}q(t) = \frac{q_{k+1} - q_k}{\tau}$$

time step



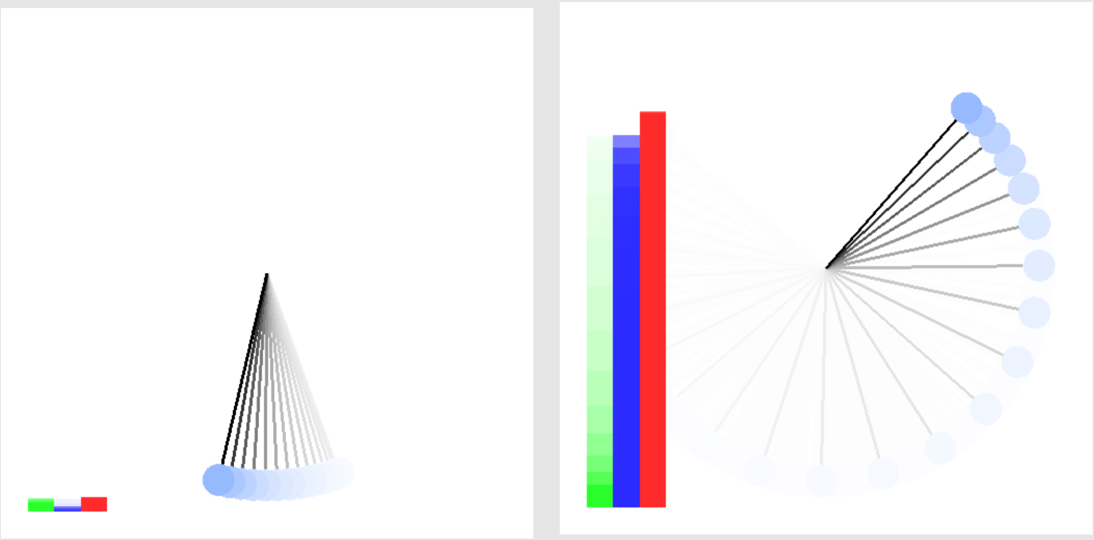
Deriving Forward & Backward Euler (2022) Steve Brunton

Forward Euler

- **Idea:** evaluate velocity at current configuration
- New configuration can then be written explicitly in terms of known data:

$$q_{k+1} = q_k + \tau * v(q_k)$$

- Very intuitive: walk a tiny bit in the direction of the velocity



starts slow

gradually moves faster

Where did all this energy come from?

Forward Euler Analysis

Let's consider behavior of forward Euler for a simple linear ODE:

$$\dot{q} = -aq, \qquad a > 0$$

q should decay over time (loss of energy to global system).
Forward Euler approximation is:

$$q_{k+1} = q_k - \tau a q_k$$

$$q_{k+1} = (1 - \tau a) q_k$$

Which means after n steps, we have:

$$q_n = (1 - \tau a)^n q_0$$

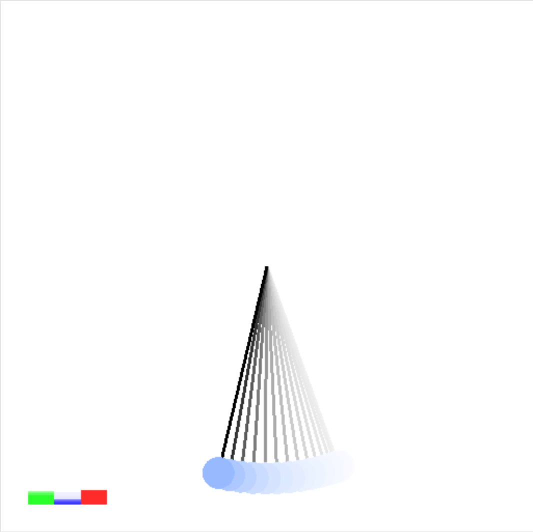
Decays only if $|1 - \tau a| < 1$, or equivalently, if $\tau < 2/a$
In practice: need very small time steps if a is large

Backward Euler

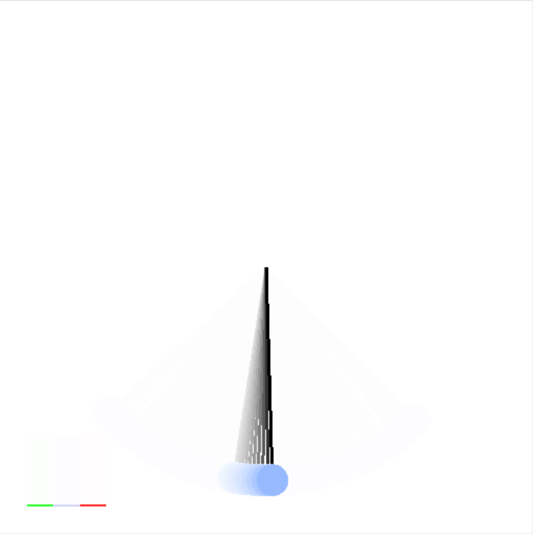
- **Idea:** evaluate velocity at next configuration
- New configuration implicit, output depends on input:

$$q_{k+1} = q_k + \tau * v(q_{k+1})$$

- Much harder to solve, since in general v can be very nonlinear!
  - More of a constraint problem: find a q_{k+1} that satisfies the above equation
  - Generally expensive to solve



starts slow



gradually slows down

Where did all this energy go?

Backward Euler Analysis

Again, let's consider a simple linear ODE:

$$\dot{q} = -aq, \qquad a > 0$$

q should decay over time (loss of energy to global system).

Backward Euler approximation is:

$$(q_{k+1} - q_k)/\tau = -aq_{k+1}$$

$$\frac{q_{k+1}}{\tau} + aq_{k+1} = \frac{q_k}{\tau}$$

$$(1 + \tau a)q_{k+1} = q_k$$

$$q_{k+1} = \frac{1}{1+\tau a} q_k$$

Which means after n steps, we have:

$$q_n = \left(\frac{1}{1+\tau a}\right)^n q_0$$

Decays only if $|1 + \tau a| > 1$, which is always true!

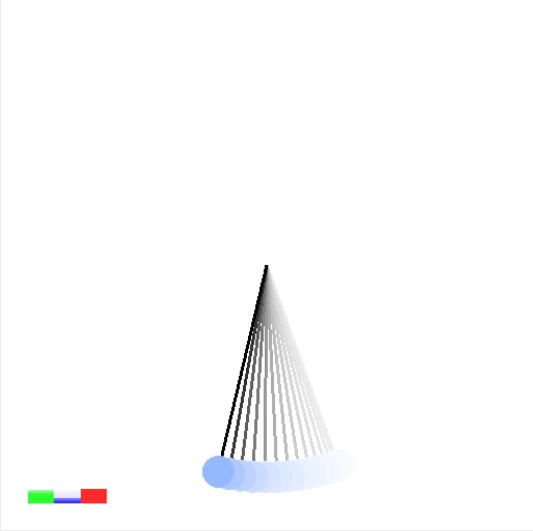
Backwards Euler is **unconditionally stable** for linear ODEs!

Symplectic Euler

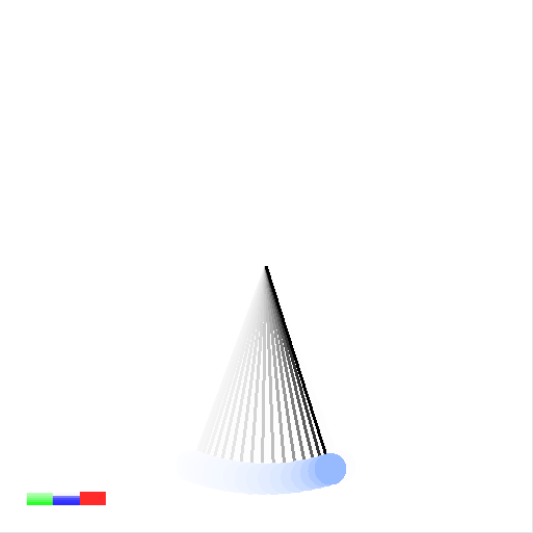
- Nice alternative is Symplectic Euler
  - Update velocity using current configuration q_k
  - Update configuration using new velocity v_{k+1}

$$v_{k+1} = v_k + \tau * a(q_k)$$
$$q_{k+1} = q_k + \tau * v_{k+1}$$

- Pendulum now conserves energy *almost exactly*, forever



starts slow



keeps on ticking

Proof? The analysis isn't very easy...

Explicit Euler Methods

[Forward]

$$v_{k+1} = v_k + \tau * a(q_k)$$
$$q_{k+1} = q_k + \tau * v_k$$

[Symplectic]

$$v_{k+1} = v_k + \tau * a(q_k)$$
$$q_{k+1} = q_k + \tau * v_{k+1}$$

[Leapfrog]

$$v_{k+1} = v_{k+0.5} + \frac{\tau}{2} * a(q_k)$$
$$q_{k+1} = q_k + \tau * v_{k+1}$$
$$v_{k+1.5} = v_{k+1} + \frac{\tau}{2} * a(q_k)$$

[RK2]

$$v'_{k+1} = \tau * a(q_k)$$
$$v''_{k+1} = \tau * a(q_k + \frac{v'_{k+1}}{2})$$
$$v_{k+1} = v_k + v''_{k+1}$$
$$q_{k+1} = q_k + \tau * v_{k+1}$$

[RK4]

$$v'_{k+1} = \tau * a(q_k)$$
$$v''_{k+1} = \tau * a(q_k + \frac{v'_{k+1}}{2})$$
$$v'''_{k+1} = \tau * a(q_k + \frac{v''_{k+1}}{2})$$
$$v''''_{k+1} = \tau * a(q_k + v'''_{k+1})$$
$$q_{k+1} = q_k + \frac{1}{6}(v'_{k+1} + 2v''_{k+1} + 2v'''_{k+1} + v''''_{k+1})$$