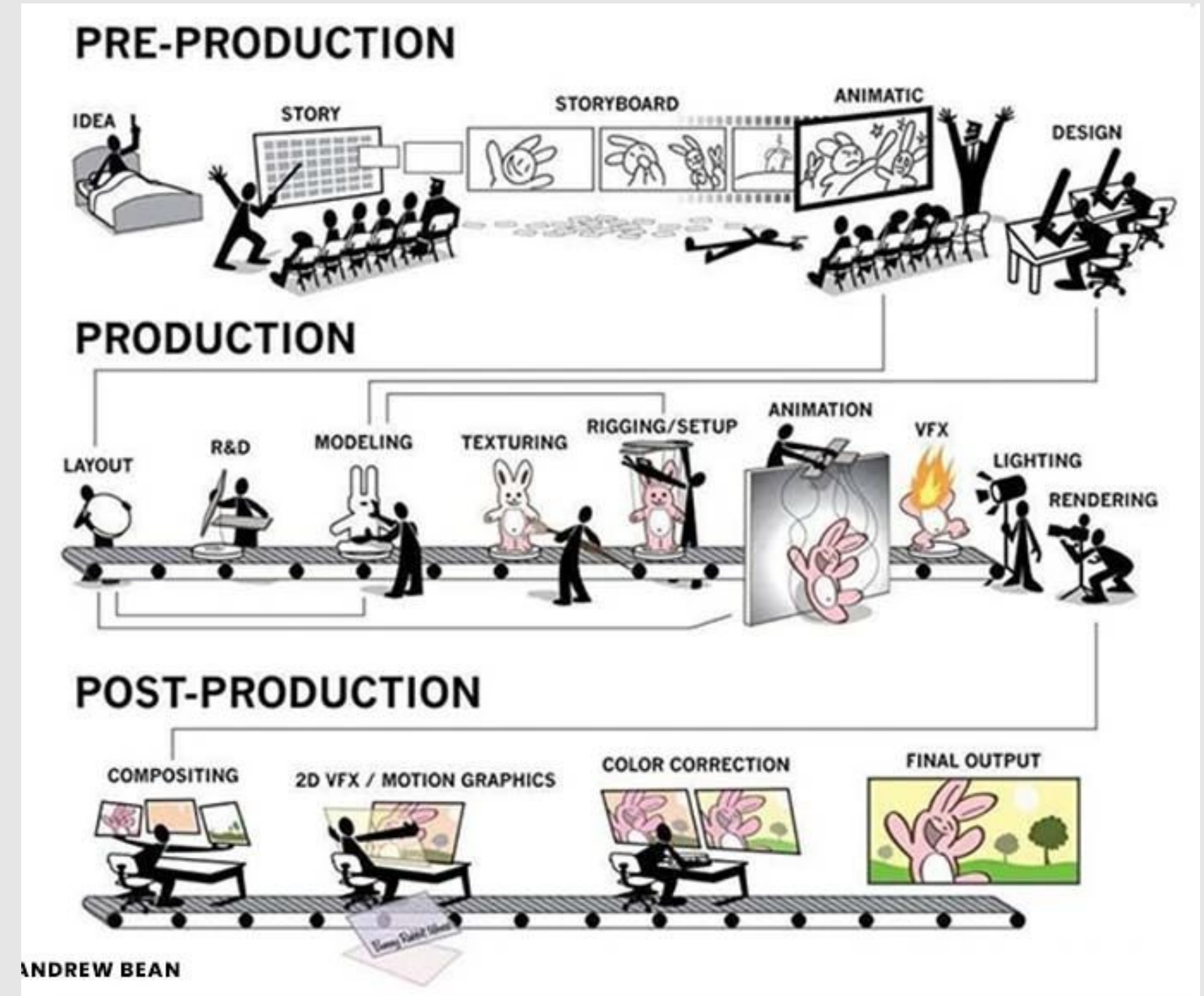
A4: Animation

Welcome to Animation

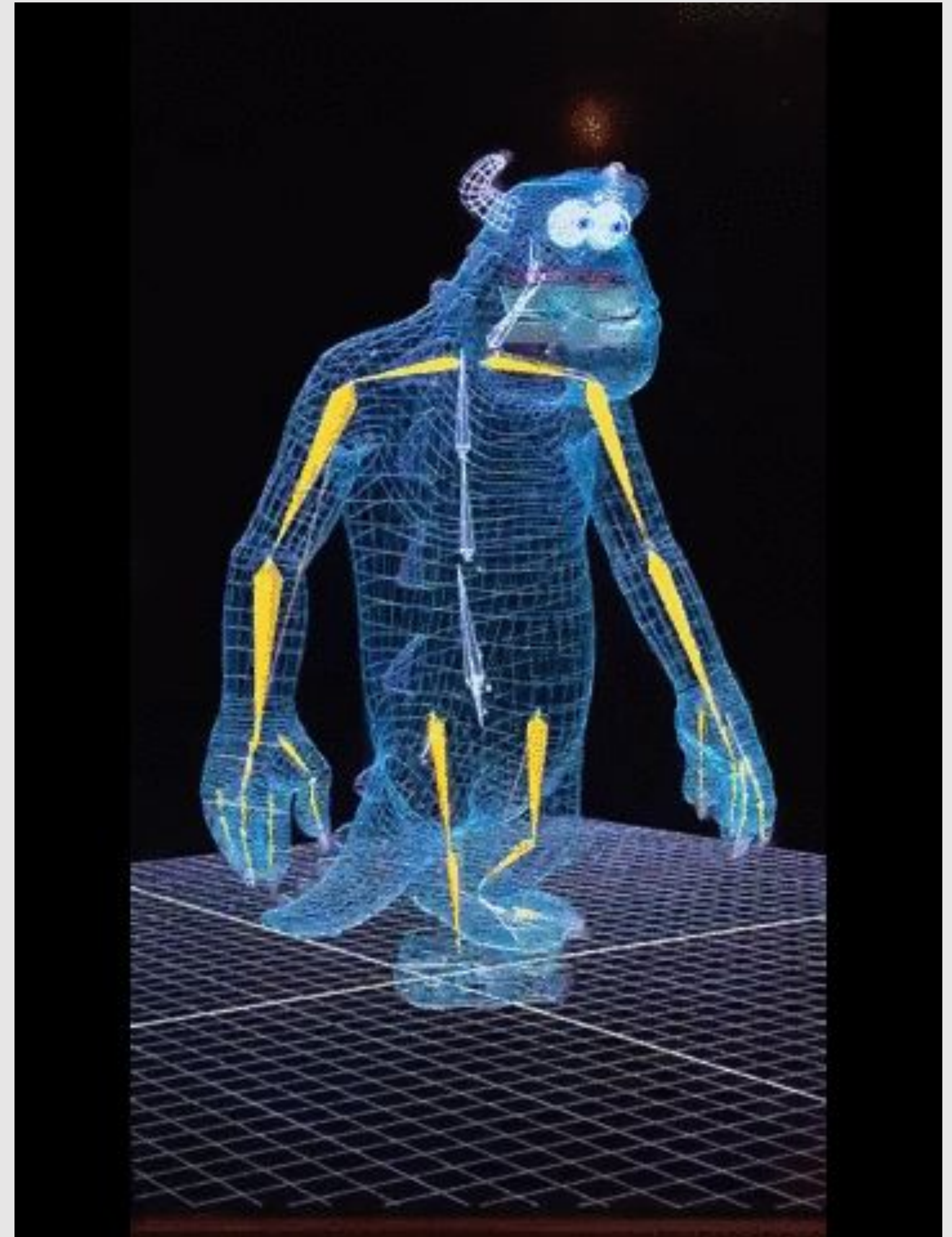
- Want to create photorealistic, fluid and exciting animations without drawing out every image
- Enter computer animation:
  - Create well-defined character models and meshes
  - Set keyframes using kinematics
  - Interpolate between keyframes with splines
  - Use a photorealistic renderer for final results



Real world applications…

Snow castle

©Disney

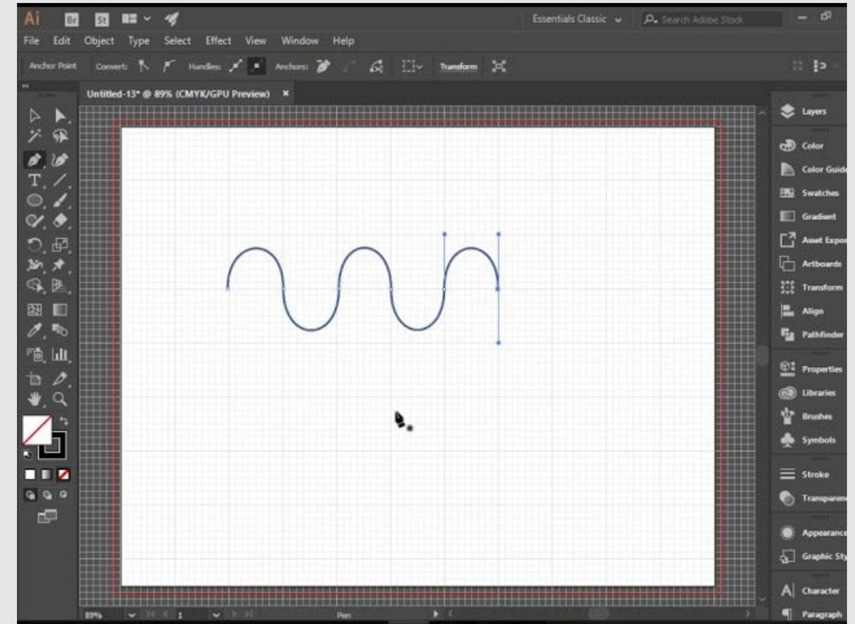


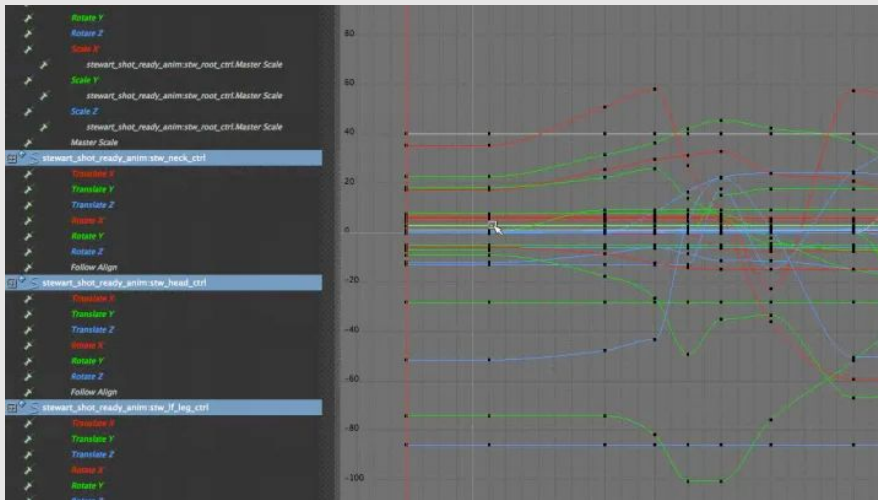
- Spline Interpolation
- Skeleton Kinematics

- Spline Interpolation
- Skeleton Kinematics

Splines Are Everywhere

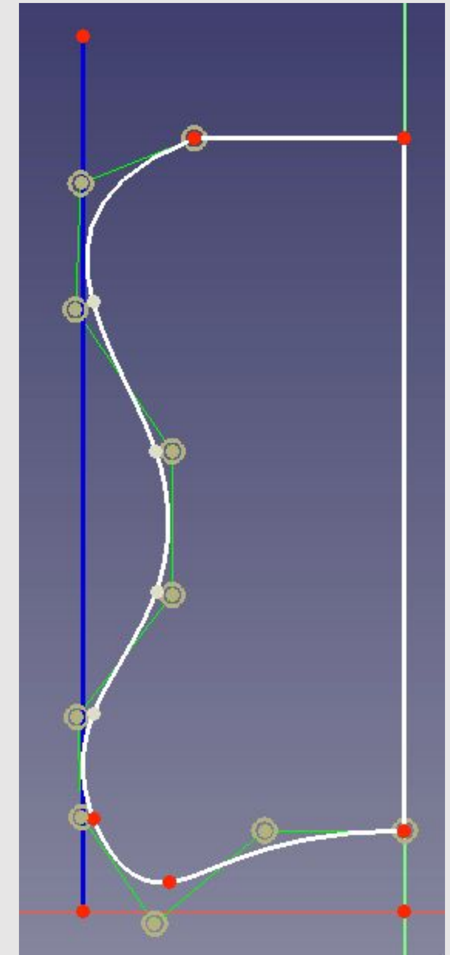
- Splines are used in many parts of the animation pipeline
  - Can be used "literally" in designing assets
  - Or can also be used to describe motion of objects when animating
  - And way more…

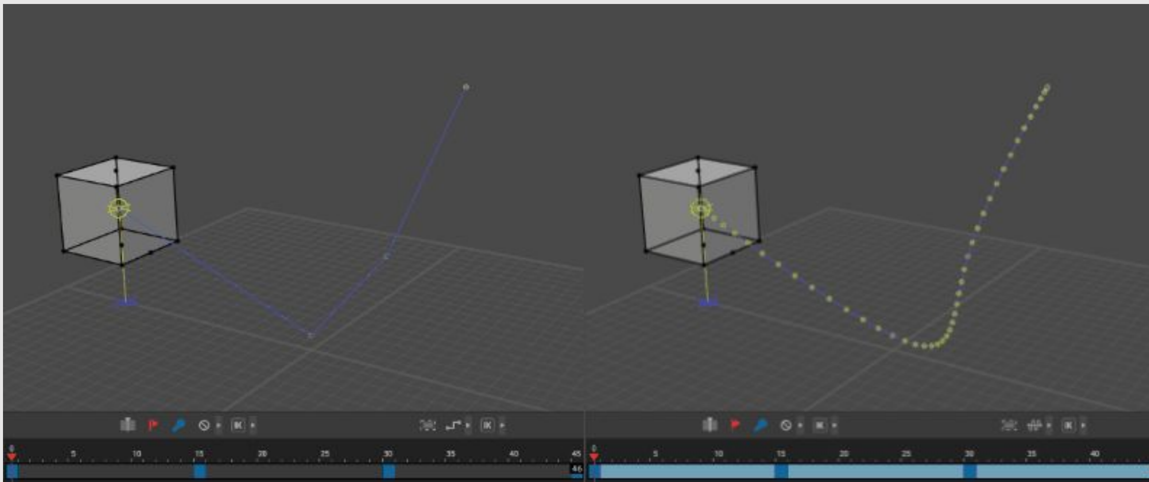




What Are Splines?

- Splines are **piecewise functions** described by polynomials for each piece
- Think of them as several curves that are connected at their **endpoints**
- Each of these curves can be modified individually without affecting the other curves, as long as the endpoints are still connected





Hermite Splines

- Each cubic piece specified by endpoints and tangents
  - Keyframes set at endpoints:

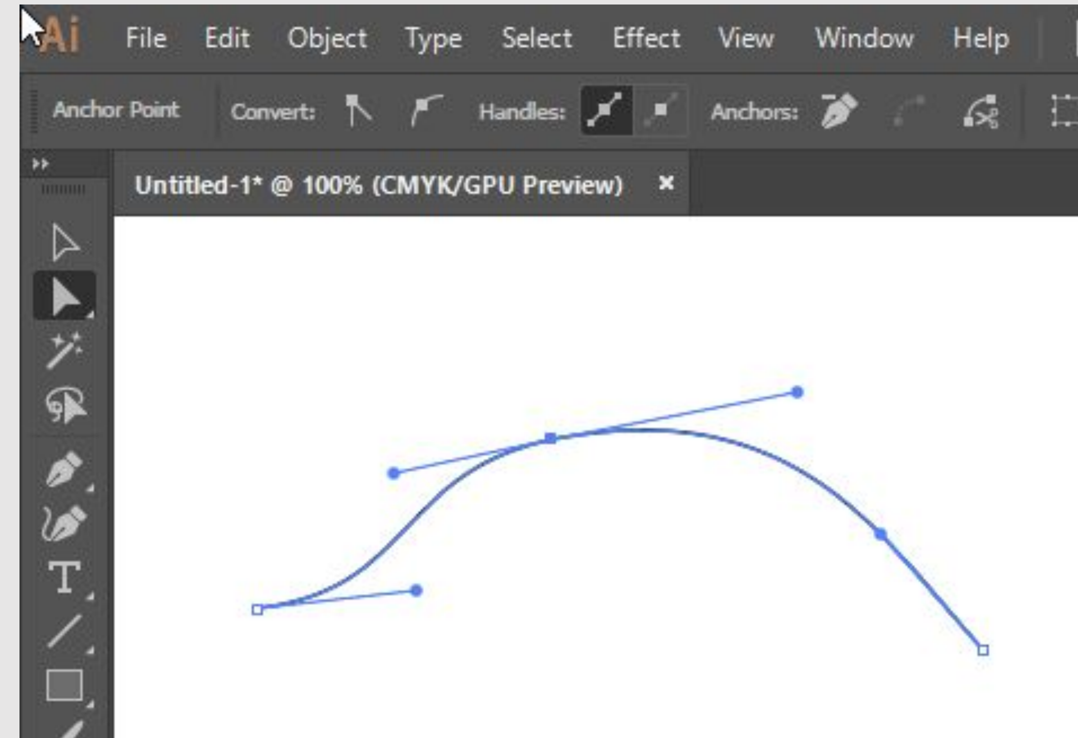
$$p_i(0) = f_i, \qquad p_i(1) = f_{i+1}, \qquad \forall i = 0, \dots, n-1$$

  - Tangents set at endpoint:

$$p'_i(0) = u_i, \qquad p'_i(1) = u_{i+1}, \qquad \forall i = 0, \dots, n-1$$

- Total equations:
  - 2n + 2n = 4n

- Commonly used in vector art programs
  - Illustrator
  - Inkscape
  - SVGs



Hermite Splines

Hermite form given by:

$$p(t) = h_{00}(t)p_0 + h_{10}(t)m_0 + h_{01}(t)p_1 + h_{11}(t)m_1$$

where:

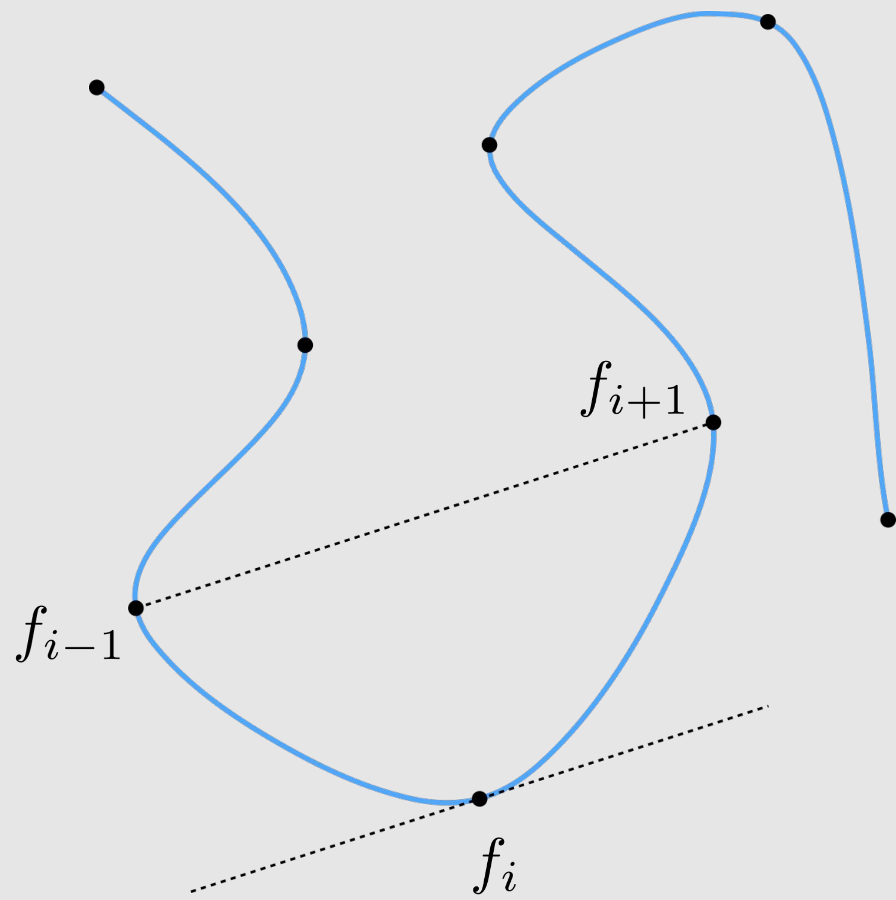
- p_0, p_1 are endpoints
- m_0, m_1 are endpoint tangents (derivatives)
- h_ij(t) are cubic hermite basis functions

Catmull-Rom Splines

- A specialized version of Hermite splines
  - Only specify keyframes
  - Tangents computed as:

$$u_i := \frac{f_{i+1} - f_{i-1}}{t_{i+1} - t_{i-1}}$$

- All the same properties of Hermite splines
- Commonly used in motion capture
  - When we have tracking data, but no tangent data
  - Easy to generate tangent data



Catmull-Rom Splines

- Construct tangents via finite differences:

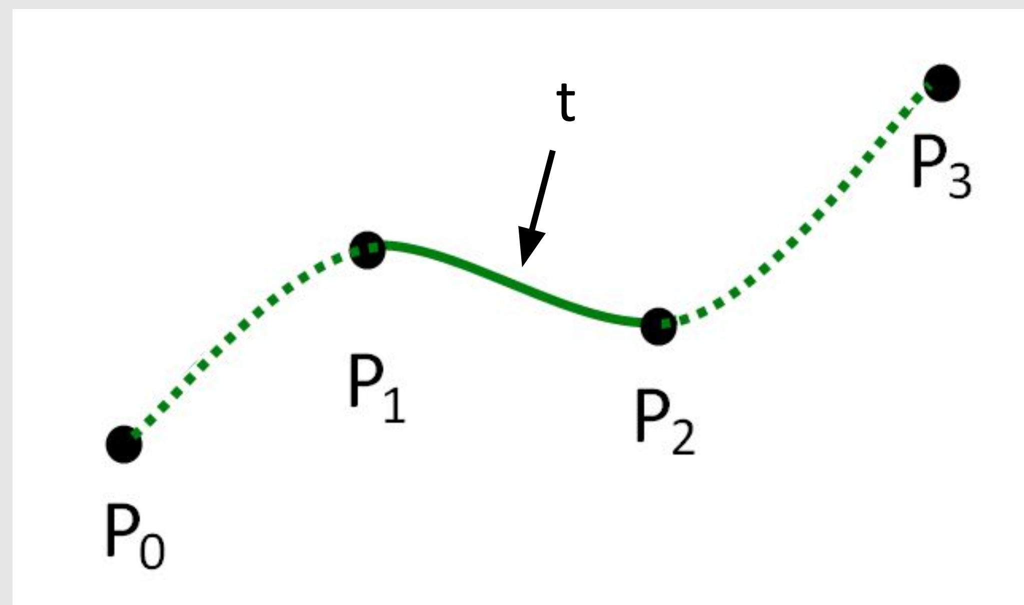
$$u_1 = \frac{p_2 - p_0}{t_2 - t_0} \qquad u_2 = \frac{p_3 - p_1}{t_3 - t_1}$$

- Perform cubic interpolation on splines:

$$p(t) = h_{00}(t) * p_1 + h_{10}(t) * u_1 + h_{01}(t) * p_2 + h_{11}(t) * u_2$$

$$h_{00}(t) = 2t^3 - 3t^2 + 1$$
$$h_{10}(t) = t^3 - 2t^2 + t$$
$$h_{01}(t) = -2t^3 + 3t^2$$
$$h_{11}(t) = t^3 - t^2$$

Cubic interpolation is piecewise!
Must normalize t to range [0,1]



Catmull-Rom Splines

- Need to normalize time

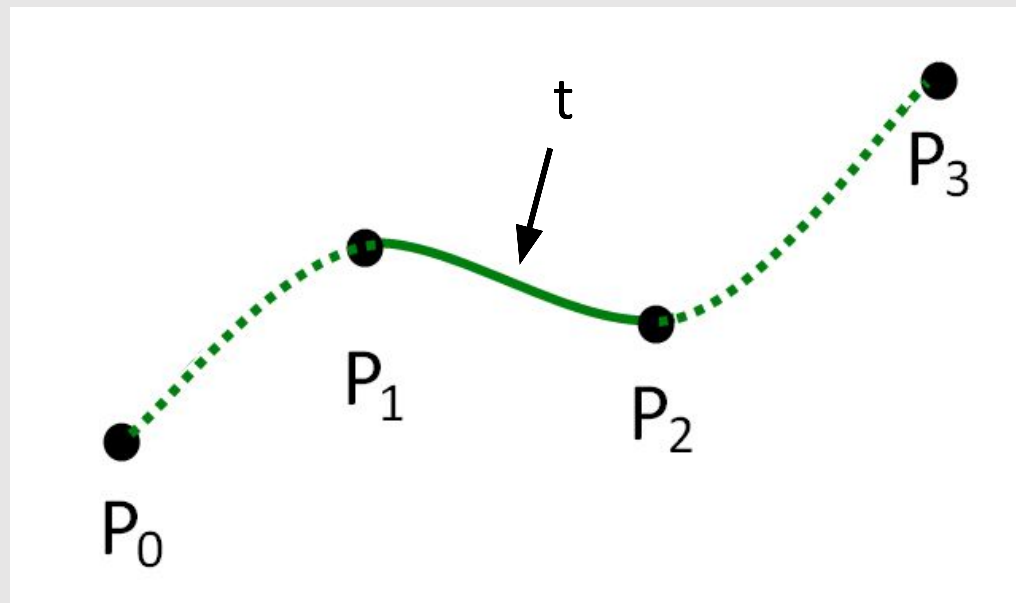
$$t_{norm} = \frac{t - t_1}{t_2 - t_1}$$

- Need to normalize tangents as well

$$u_1 = \frac{p_2 - p_0}{\frac{t_2 - t_1}{t_2 - t_1} - \frac{t_0 - t_1}{t_2 - t_1}}$$

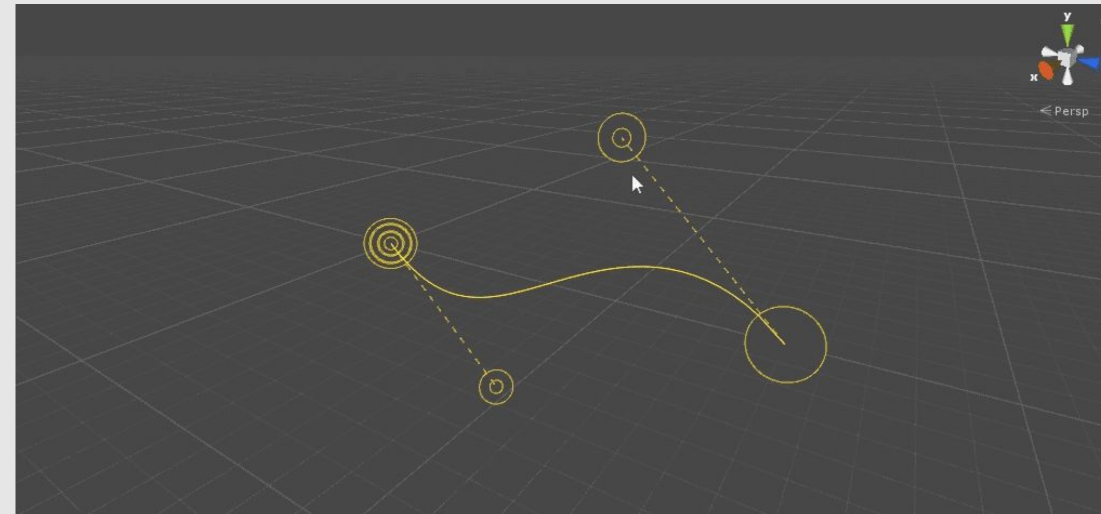
$$u_1 = \frac{p_2 - p_0}{(t_2 - t_1) - (t_0 - t_1)} * (t_2 - t_1)$$

$$u_1 = \frac{p_2 - p_0}{t_2 - t_0} * (t_2 - t_1)$$



Three Main Spline Properties

- Interpolation:
  - Does the spline pass through the control points you specified?
- Continuity:
  - C0: Are the keyframes continuous?
  - C1: Are the first derivatives continuous?
  - C2: Are the second derivatives continuous?
- Locality:
  - Changing one point / part of the curve does not change the entire curve

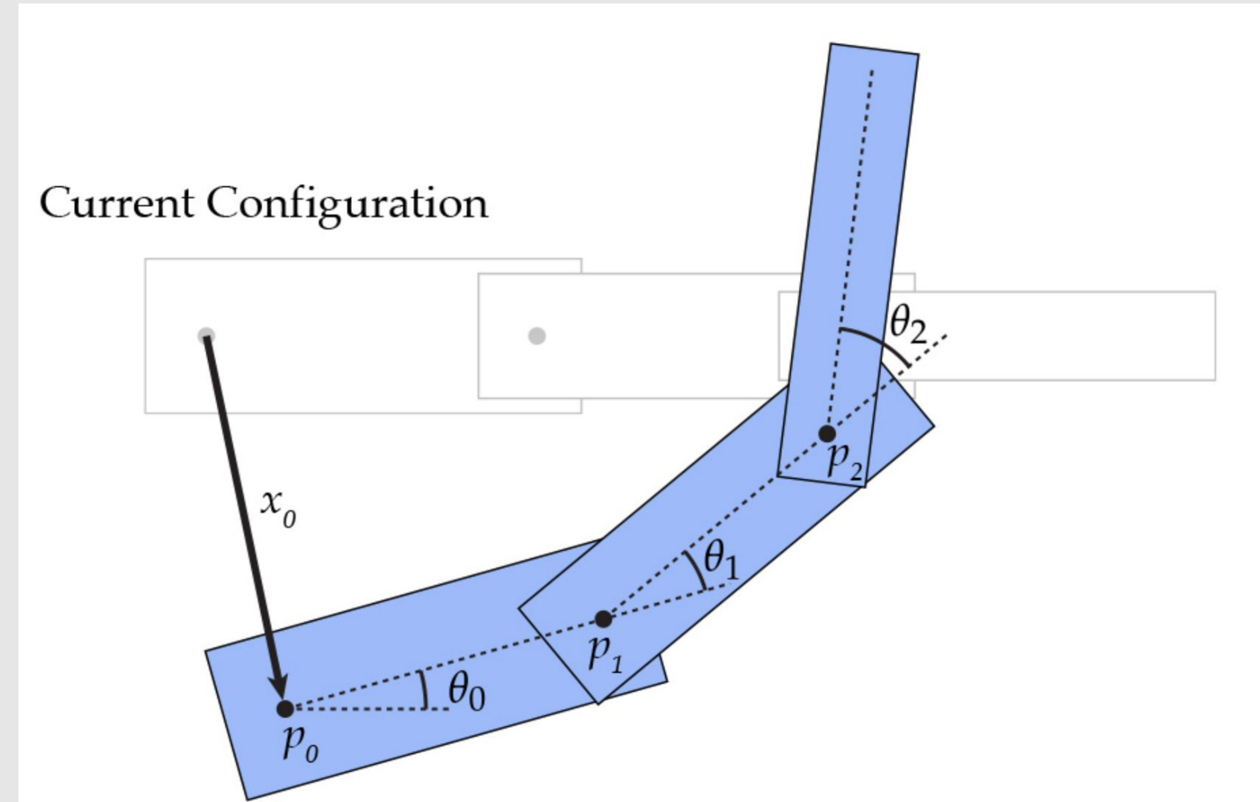


- Spline Interpolation
- Skeleton Kinematics

Forward Kinematics

FK is all about calculating the overall transformation for a bone. The core idea is that the transformation applied to a **parent joint** is also applied to all of its **children** joints. This results in a *chain* of transformations.

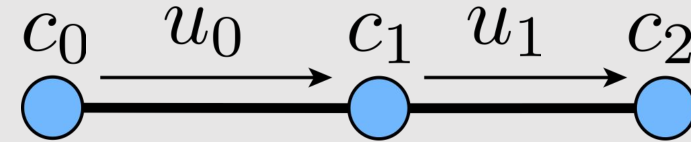
- **“Bind”** position: the rotation should be zero
- **“Posed”** position: take into account the “pose” of the joint (euler angle)



Bind vs Posed Position

Feature	**Bind Position (Bj)**	**Posed Position (Pj)**
Rotation	Should be zero (pose = (0,0,0)).	Takes into account the pose (Euler angle).
Transform	Pure translation by the parent's extent	Rotation (XYZ order) followed by translation by the parent's extent
Implementation	Skeleton::bind_pose	Skeleton::current_pose

A Note About Spaces



- Bind-to-Local:

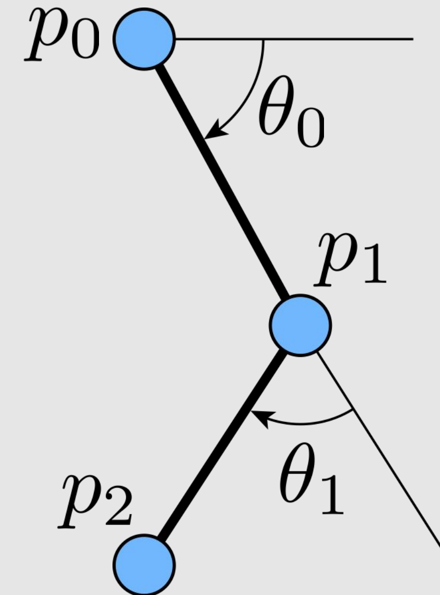
$$c_0 = T(u_0)\,T(u_1)\,c_2$$

- Pose-to-Local:

$$p_0 = R(\theta_0)\,T(u_0)\,R(\theta_1)\,T(u_1)\,R(\theta_2)\,p_2$$

need to undo p2's orientation

we give you `compute_rotation_axes` which can be used to calculate this rotation matrix



Rotations and transformations will be saved as child-to-parent

- No need to invert

Inverse Kinematics

Idea: move the skeleton towards target point using gradient descent

$$X_{k+1} = X_k - \tau \nabla f \qquad f(\theta(t)) = \frac{1}{2}|p(\theta(t)) - q|^2$$

Procedure:

- `Skeleton::gradient_in_current_pose`
  - first calculate the jacobian via $(J_\theta)_i = \vec{r} \times \vec{p}$
  - then approximate gradient using jacobian via $\nabla_\theta f \approx \alpha J_\theta^T (p(\theta) - q)$
- `Skeleton::solve_ik`
  - Use gradient descent to calculate the joints' pose at the next time step
  - See intro to optimization lecture

Inverse Kinematic Gradient

$$\frac{df}{d\theta_k^y} = \frac{d}{d\theta_k^y} \sum_{(i,h)} \frac{1}{2} |p_i(q) - h|^2$$

Take gradient with respect to function

$$\frac{df}{d\theta_k^y} = \sum_{(i,h)} (p_i(q) - h) \frac{dp_i}{d\theta_k^y}$$

Expand p_i into transformations. Each rotation in 3D is axis-aligned

$$\frac{dp_i}{d\theta_k^y} = \frac{d}{d\theta_k^y}\left[\prod_{j=0,i-1} R(\theta_j^z)R\left(\theta_j^y\right)R(\theta_j^x)T(u_j)\right] R(\theta_i^z)R\left(\theta_i^y\right)R(\theta_i^x)u_i$$

Gradient breaks down into 3 parts:

$$\frac{dp_i}{d\theta_k^y} = R(\theta_0^z)R\left(\theta_0^y\right)R(\theta_0^x)T(u_0)\ldots R(\theta_k^z)\frac{d}{d\theta_k^y}R\left(\theta_k^y\right)R(\theta_k^x)T(u_i)\ldots R(\theta_i^z)R\left(\theta_i^y\right)R(\theta_i^x)u_i$$

[linear transformation] [derivative] [transformed point]

Inverse Kinematic Gradient

$$\frac{dp_i}{d\theta_k^y} = ???$$

Fun fact: by transforming the axis of rotation and base point to local coordinates, Then the derivative of the rotation $R(\theta_k^y)$ by amount θ_k^y around axis y and center r of point p becomes:

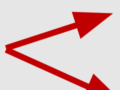
$$\frac{dp_i}{d\theta_k^y} = y \times (p - r)$$

constant for a given handle

$p =$ [linear transformation] $[R(\theta_k^y)]$ [transformed point]

$r =$ [linear transformation'] [0,0,0]

$y =$ ([linear transformation'] $[R(\theta_k^z)]$).rotate(θ_k^y)



specific to the current joint

[linear transformation'] = all rotations and transformations up to, but not including the kth bone

Inverse Kinematic Gradient

```
vec3 gradient_in_current_pose() {

  for (auto &handle : handles) {

    Vec3 h = handle.target;
    Vec3 p = // TODO: compute output point

    // walk up the kinematic chain
    for (BoneIndex b = handle.bone; b < bones.size(); b = bones[b].parent) {
      Bone const &bone = bones[b];
      Mat4 xf = // TODO: compute [linear transform']

      Vec3 r = xf * Vec3{0.0f, 0.0f, 0.0f};

      Vec3 x = // TODO: compute bone's x-axis in local space
      Vec3 y = // TODO: compute bone's y-axis in local space
      Vec3 z = // TODO: compute bone's z-axis in local space

      gradient[b].x += dot(cross(x, p - r), p - h);
      gradient[b].y += dot(cross(y, p - r), p - h);
      gradient[b].z += dot(cross(z, p - r), p - h);
    }
  }
}
```

Inverse Kinematic Gradient Descent

Steps:

- Call `gradient_in_current_pose()` to compute d loss / d pose
- Update positions of all the bones by the computed gradients
- Loop through each handle and calculate the loss
  - loss is $\sum_{(i,h)} \frac{1}{2} |p_i(q) - h|^2$
- If at a local minimum (e.g., gradient is near-zero), return 'true'
- If run through all steps, return `false`