Coordinate Spaces & Transformations

Wrapup from Wednesday – References

- We can declare functions that automatically cast variables to ref on input and return
- This does not mean that the variable accepting the return must also be a reference. Consider how `h` is being assigned a reference even though it is not a reference type itself
- So then why bother returning a ref?
  - Sometimes we just want reassurance that we're always returning a reference to the same vertex halfedge in our call to `vert->halfedge()`, and that no duplicates are being created

```
class Vertex {
public:
    HalfedgeRef& halfedge() {return _halfedge;}
    ...
private:
    ...
    HalfedgeRef _halfedge;
};
```

```
float totalArea = 0.0f;
// HalfedgeRef type accepts HalfedgeRef&
HalfEdgeRef h = vert->halfedge(); do {
  if(!h->face()->is_boundary()) {
    totalArea += h->face()->area();
  }
  // because it isn't a ref, we can keep updating it
  h = h->twin()->next();
}
while(h != vert->halfedge());
```

- The Rasterization Pipeline
- Transformations
- Homogeneous Coordinates
- 3D Rotations

The Goal Of Graphics

- Render very high complexity 3D scenes
  - Hundreds of thousands to millions to billions of triangles in a scene
  - Complex vertex and fragment shader computations
  - High resolution screen outputs (~10Mpixel + supersampling)
  - 30-120 fps

- Limited hardware resources
  - Can't always afford an RTX 4090
  - Be efficient enough to run on commercial hardware

Unreal Engine 5 Tech Demo (2020) Epic Games

Processing The Graphics Pipeline

- Modern real time image generation based on rasterization
- **INPUT:**
  - 3D “primitives”—essentially all triangles!
  - Colors
  - Textures
- **OUTPUT:**
  - Bitmap image (possibly w/ depth, alpha, …)

INPUT (TRIANGLES)

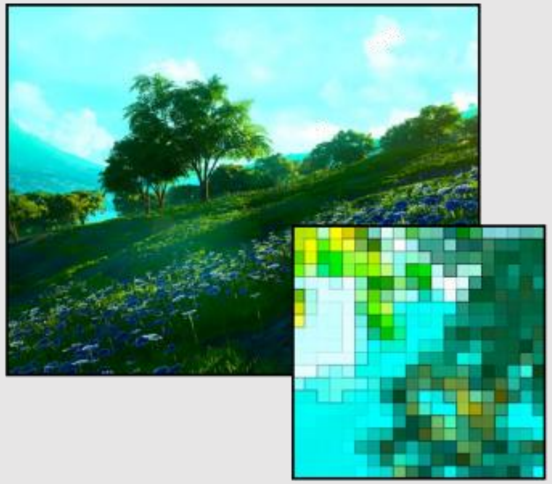
```
VERTICES
A: ( 1, 1, 1 )    E: ( 1, 1,-1 )
B: (-1, 1, 1 )    F: (-1, 1,-1 )
C: ( 1,-1, 1 )    G: ( 1,-1,-1 )
D: (-1,-1, 1 )    H: (-1,-1,-1 )

TRIANGLES
EHF, GFH, FGB, CBG,
GHC, DCH, ABD, CDB,
HED, ADE, EFA, BAF
```

→

RASTERIZATION PIPELINE

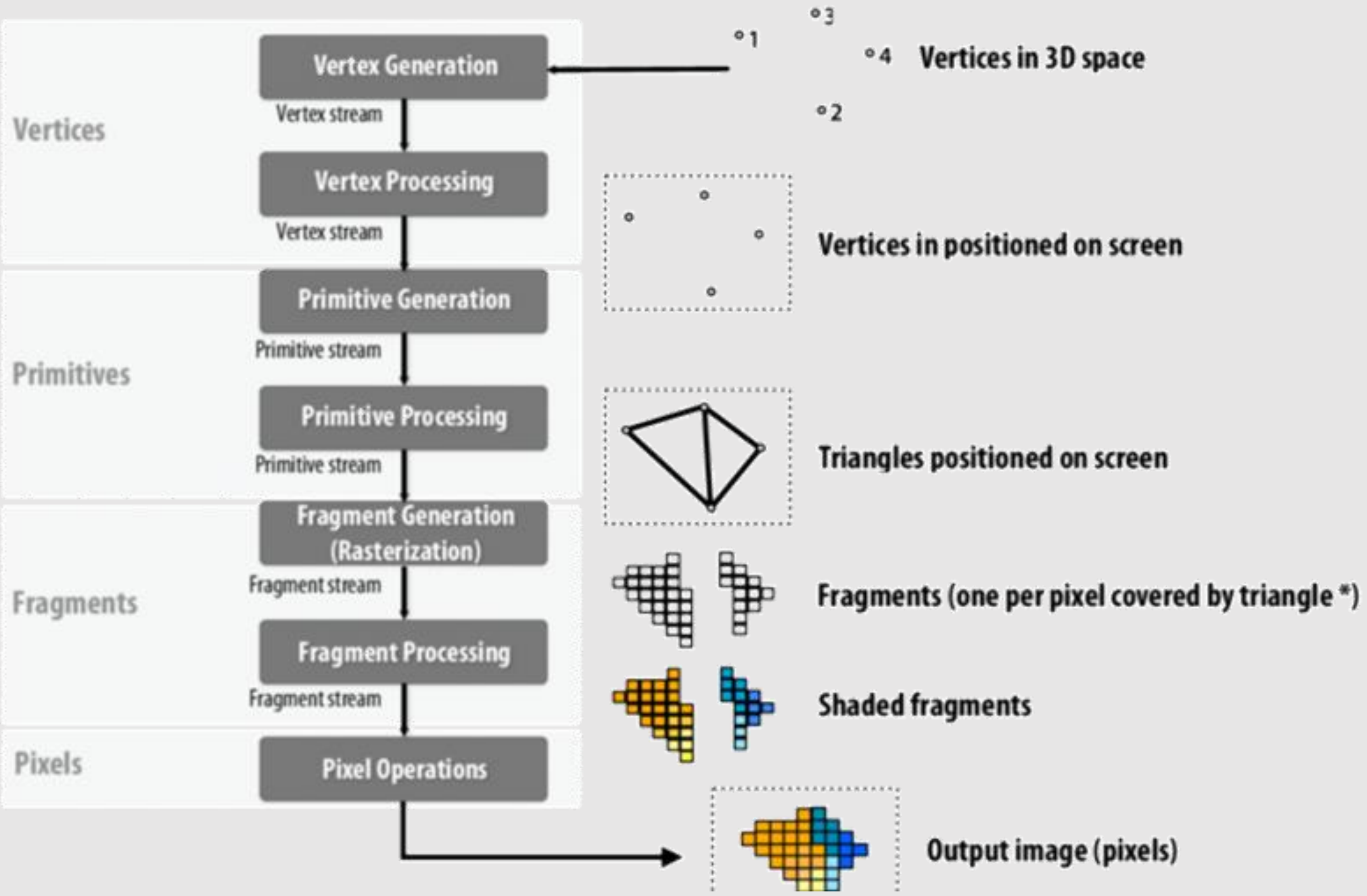
→

OUTPUT (BITMAP IMAGE)



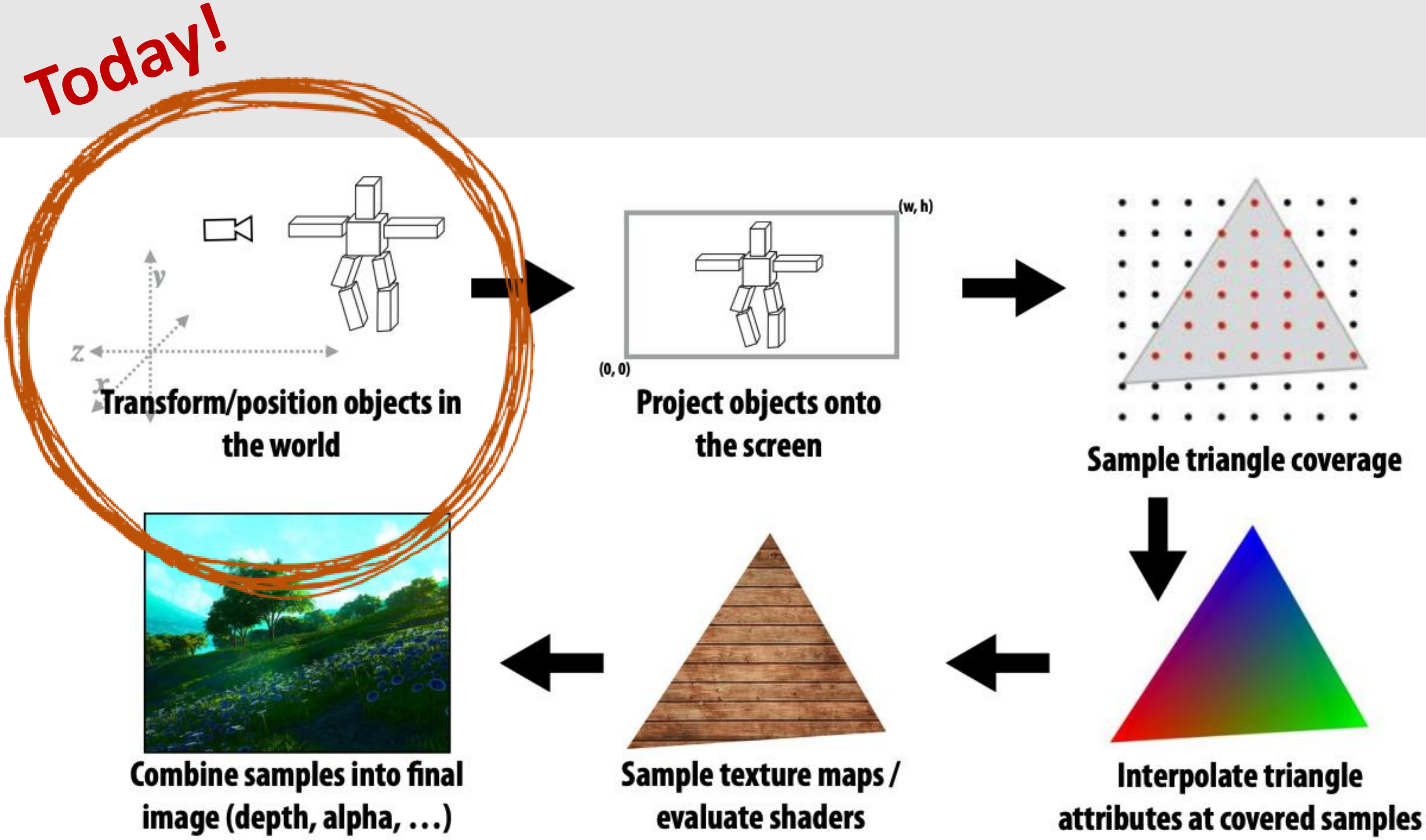
Q: How do we write software to perform rasterization?

The First Week of Class -- The Graphics Pipeline



Let’s simplify things a bit

The “Simpler” Graphics Pipeline



- ~~The Rasterization Pipeline~~
- Transformations
- Homogeneous Coordinates
- 3D Rotations

Transformations In Computer Graphics

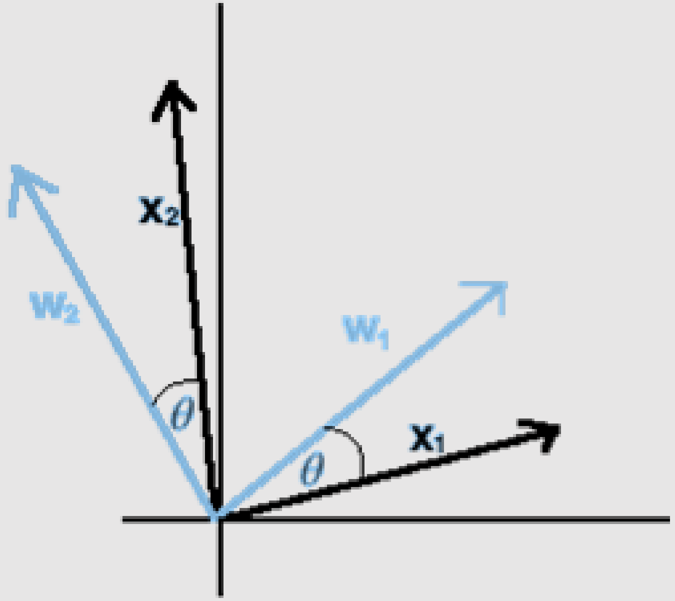
- Common uses of linear transformations:
  - Position/deform objects in space
  - Camera movements
  - Animate objects over time
  - Project 3D objects onto 2D images
  - Map 2D textures onto 3D objects
  - Project shadows of objects onto other objects

- Today we'll focus on common transformations of space (rotation, scaling, etc.) encoded by linear maps

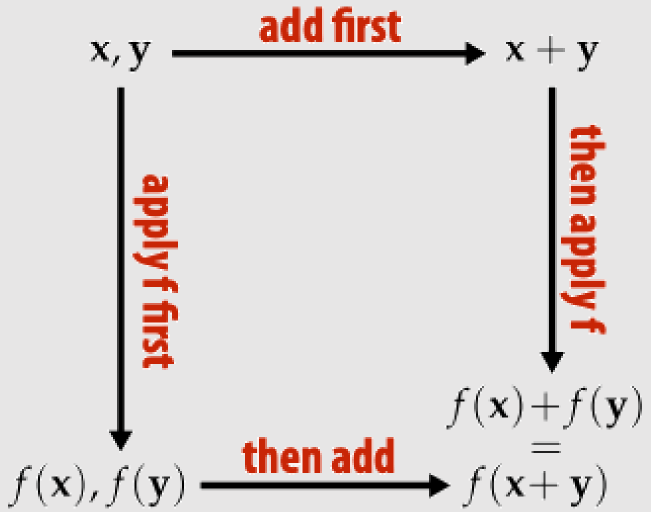
Super Mario 64: Camera Guy (1996) Nintendo

Review: Linear Maps

What does it mean for a map $f: \mathbb{R}^n \to \mathbb{R}^n$ to be linear?



Geometrically it maps lines to lines, and preserves the origin



Algebraically it preserves vector space operations (addition & scaling)

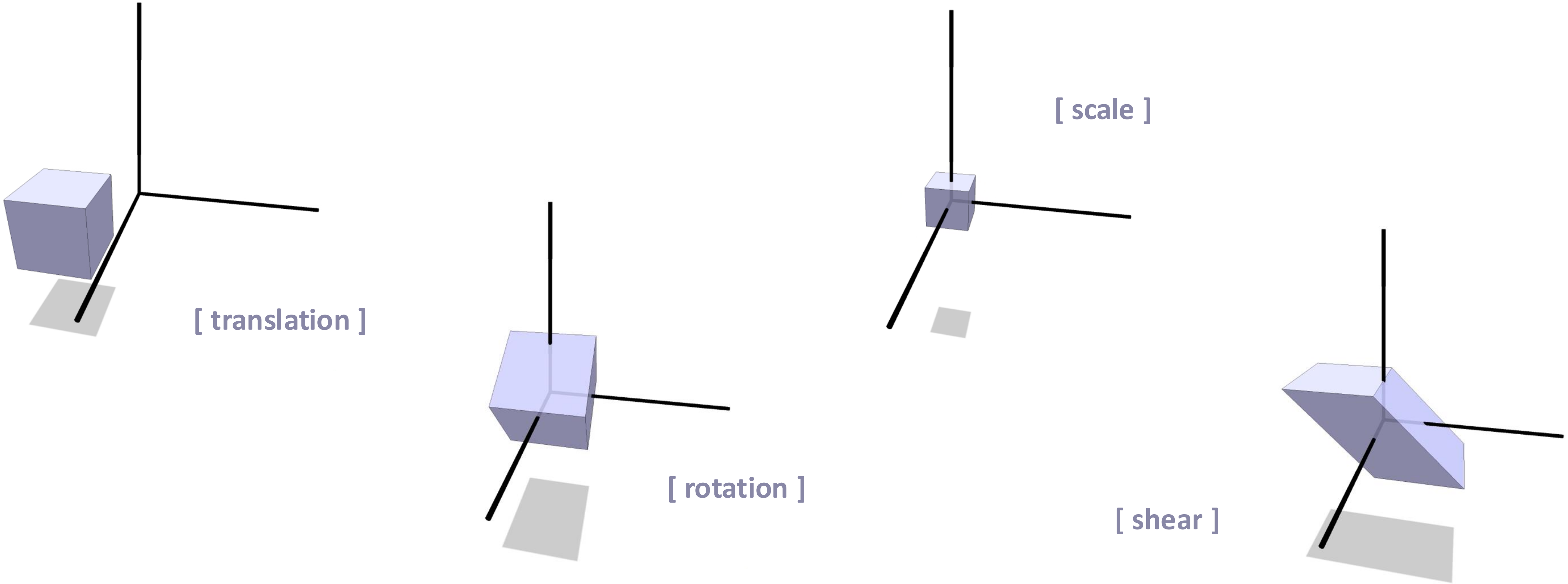
Review: Linear Maps

- Why do we care about linear transformations?
  - Cheap to apply
  - Usually pretty easy to solve for (linear systems)
  - **Composition of linear transformations is linear**
    - Product of many matrices is a single matrix
    - Gives uniform representation of transformations
    - Simplifies graphics algorithms, systems

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \cdots = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

[rotation] [scale] [rotation] [composite]

Types of Transformations

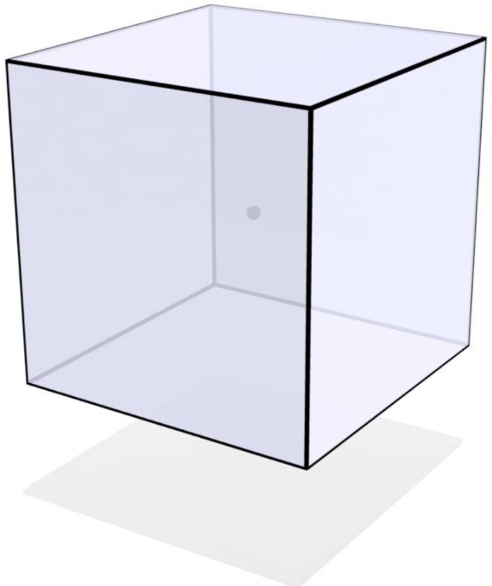


Invariants of Transformation

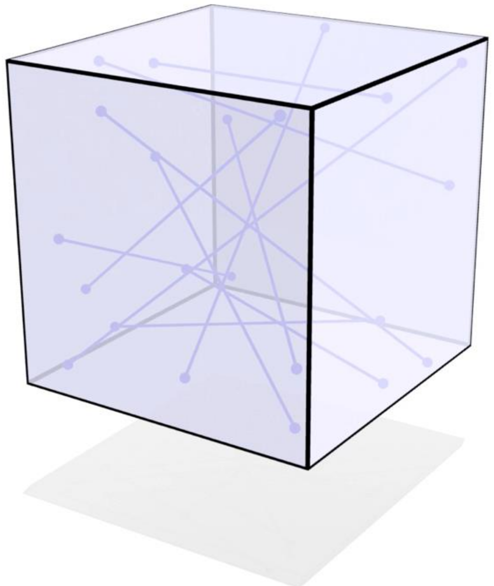
A transformation is determined by the **invariants** it preserves

transformation	**invariants**	**algebraic description**
linear	*straight lines / origin*	$f(a\mathbf{x}+\mathbf{y}) = af(\mathbf{x}) + f(\mathbf{y})$, $f(0) = 0$
translation	*differences between pairs of points*	$f(\mathbf{x}-\mathbf{y}) = \mathbf{x}-\mathbf{y}$
scaling	*lines through the origin / direction of vectors*	$f(\mathbf{x})/\lvert f(\mathbf{x})\rvert = \mathbf{x}/\lvert\mathbf{x}\rvert$
rotation	*origin / distances between points / orientation*	$\lvert f(\mathbf{x})-f(\mathbf{y})\rvert = \lvert\mathbf{x}-\mathbf{y}\rvert$, $\det(f) > 0$
…	…	…

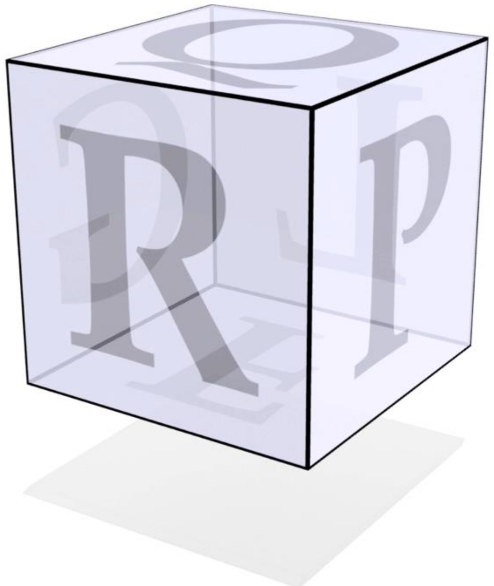
Rotation



[keeps origin fixed]



[preserves distance]



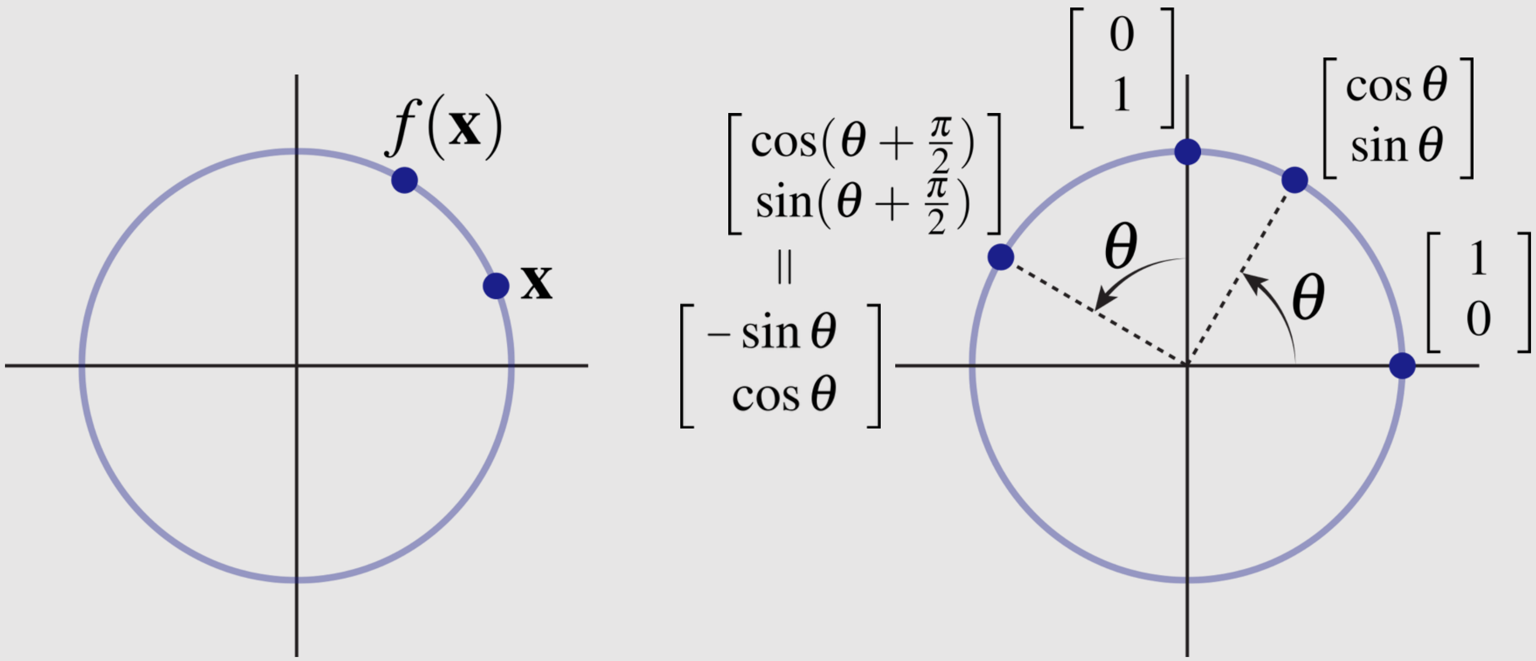
[preserves orientation]

First two properties imply rotations are **linear**

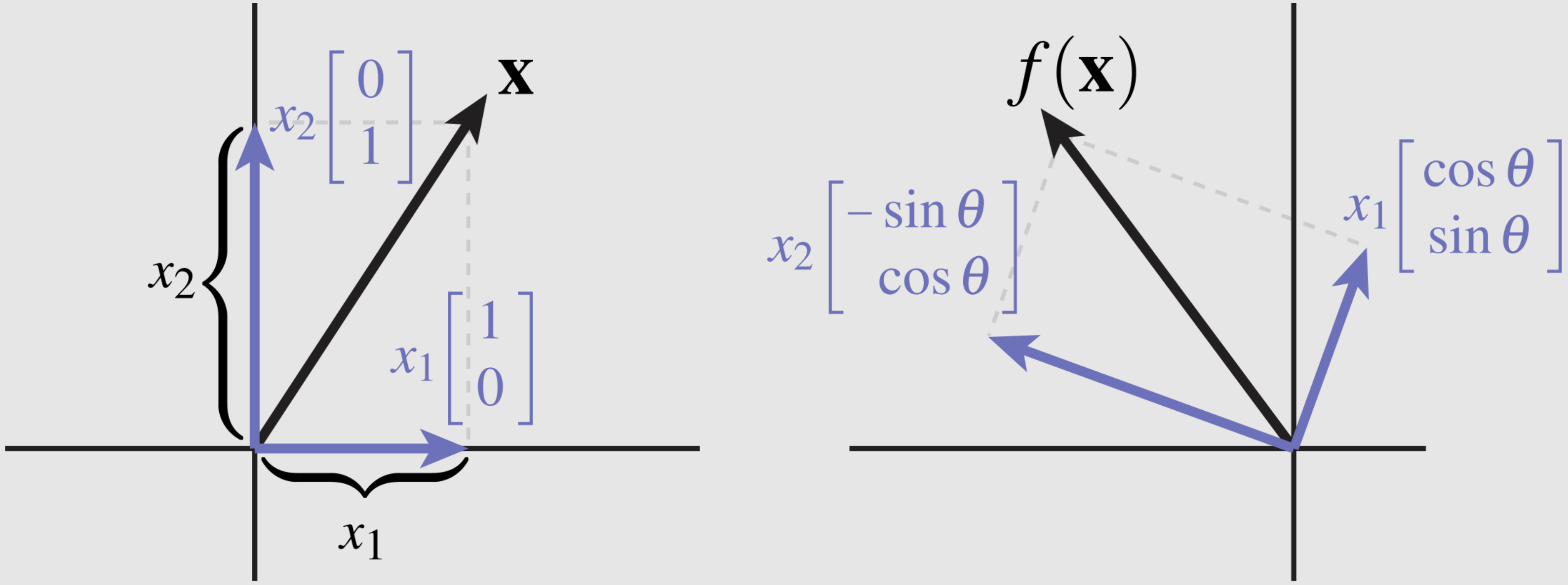
We say that a transform preserves orientation if $\det(T) > 0$

2D Rotations

Rotations preserve distances and the origin—hence, a 2D rotation by an angle θ maps each point x to a point $f(x)$ on the circle of radius $|x|$:



2D Rotations



$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$f(\mathbf{x}) = x_1 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + x_2 \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Rotations (like all transforms) are linear maps.
We can express the transform as a change of bases:

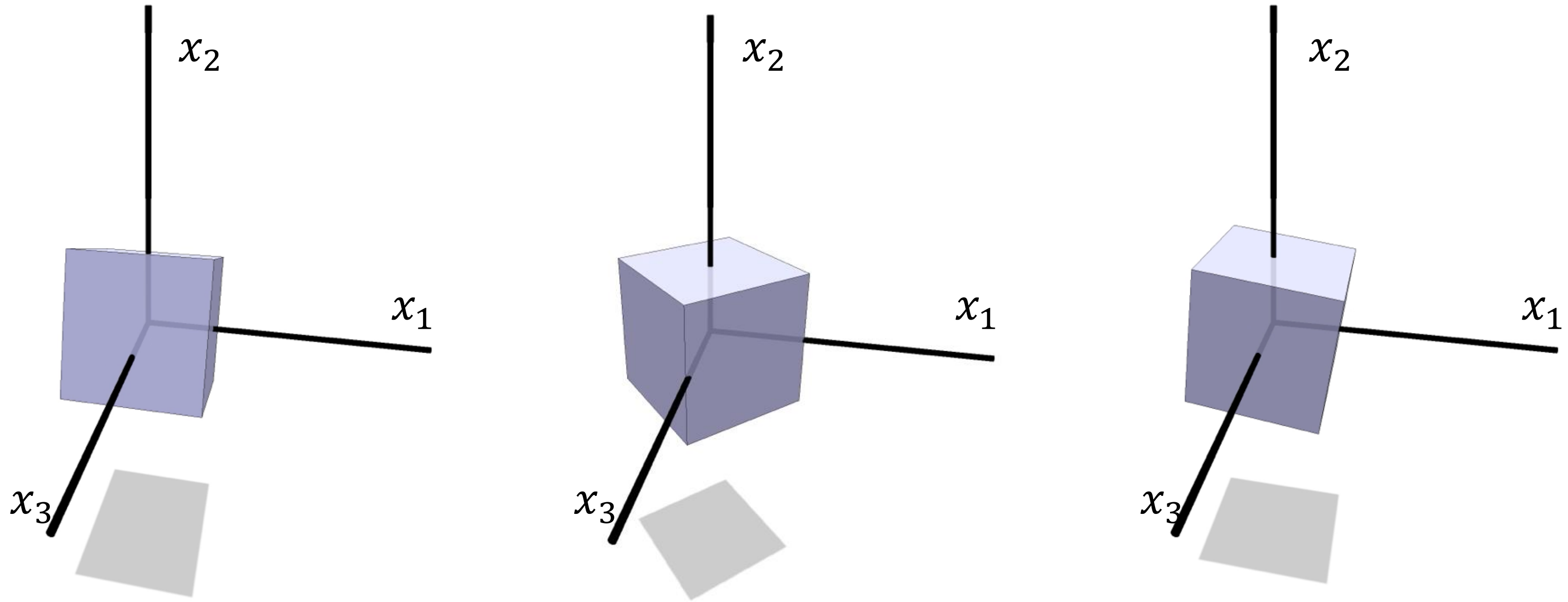
$$f_\theta(\mathbf{x}) = \begin{bmatrix} \cos\theta & -\sin(\theta) \\ \sin\theta & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3D Rotations

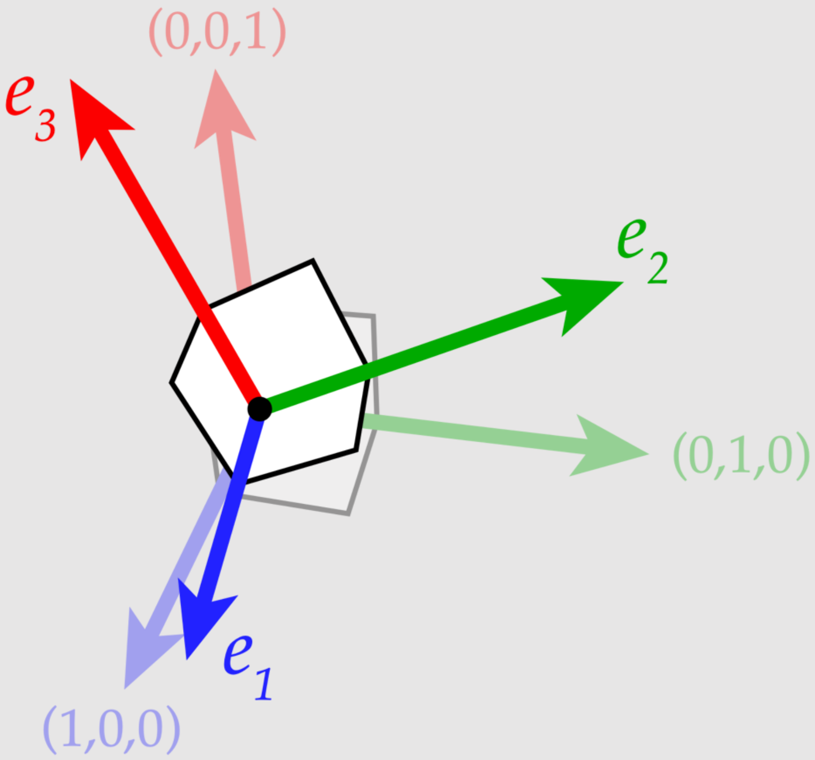
In 3D, keep one axis fixed and rotate the other two:

[rotate around x_1] **[rotate around x_2]** **[rotate around x_3]**

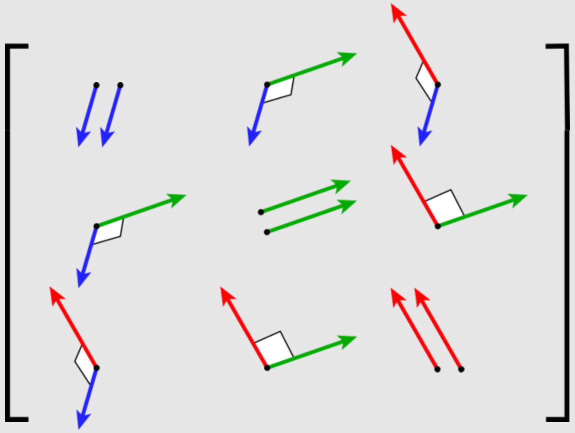
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin(\theta) \\ 0 & \sin\theta & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin(\theta) & 0 \\ \sin\theta & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



3D Inverse Rotations



$$R^{\mathsf{T}} R = \begin{bmatrix} — e_1^T — \\ — e_2^T — \\ — e_3^T — \end{bmatrix} \begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix}$$

$=$  $= \begin{bmatrix} e_1^T e_1 & e_1^T e_2 & e_1^T e_3 \\ e_2^T e_1 & e_2^T e_2 & e_2^T e_3 \\ e_3^T e_1 & e_3^T e_2 & e_3^T e_3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T R = I \quad \Rightarrow \quad R^T = R^{-1}$$

Reflections

- Does every matrix $Q^{\mathsf{T}}Q = I$ represent a rotation?
  - Must preserve:
    - Origin
    - Distance
    - Orientation

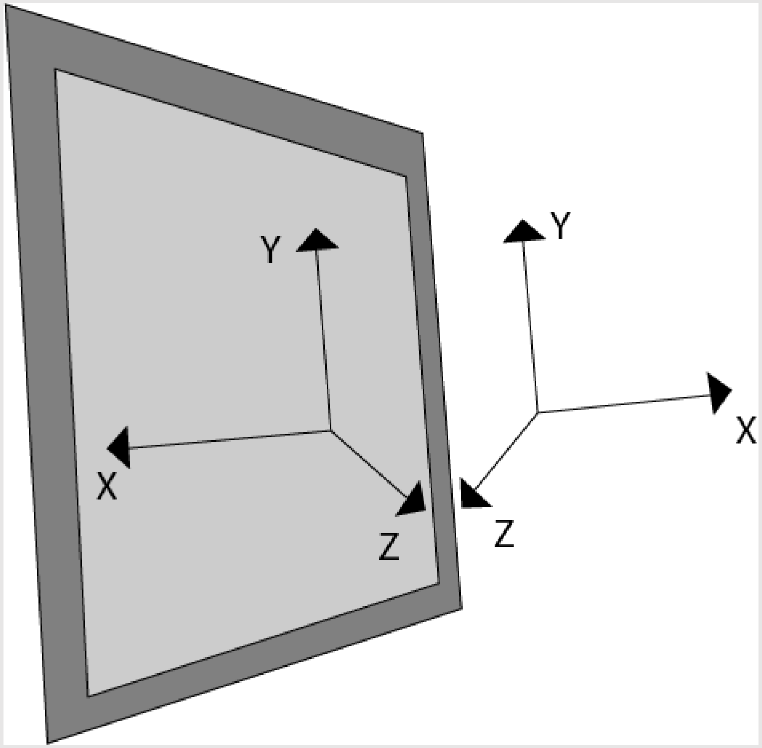
- Consider:

$$Q = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Just like rotations, Q has nice inverse properties:

$$Q^{\mathsf{T}}Q = \begin{bmatrix} (-1)^2 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- But the determinant is **negative!**
  - Not orientation preserving



Scaling

- Each vector u gets scaled by some scalar a

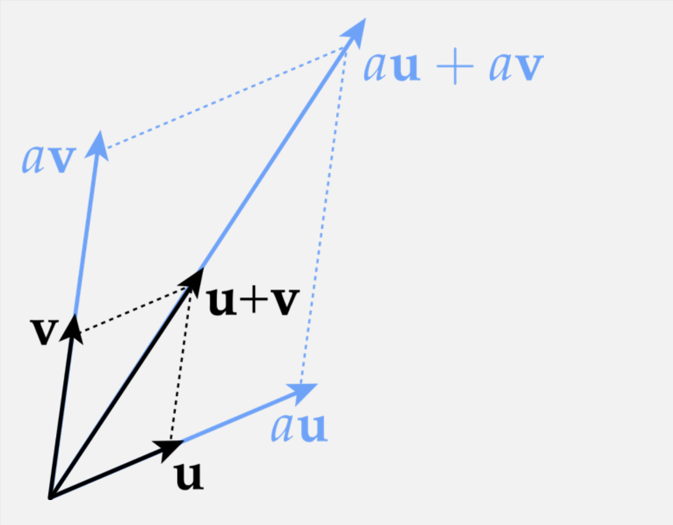
$$f(\mathbf{u}) = a\mathbf{u}, a \in \mathbb{R}$$

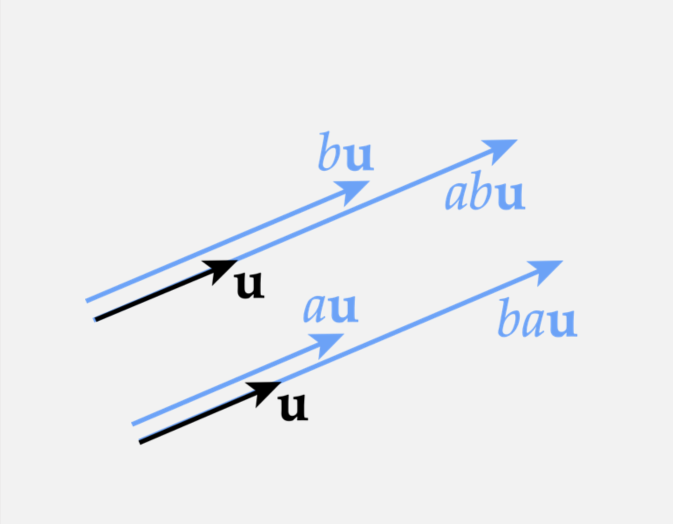
- Scaling is a linear transformation
  - Addition:

$$f(b\mathbf{u}) = ab\mathbf{u} = ba\mathbf{u} = bf(\mathbf{u})$$

  - Multiplication:

$$f(\mathbf{u}+\mathbf{v}) = \\ a(\mathbf{u}+\mathbf{v}) = \\ a\mathbf{u} + a\mathbf{v} = \\ f(\mathbf{u}) + f(\mathbf{v})$$





Negative Scaling

Can think of negative scaling as a series of reflections

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Also works in 3D:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

[flip x] [flip y] [flip z]

In 2D, two reflections so resulting $(\det(T) > 0)$
In 3D, three reflections so resulting $(\det(T) < 0)$

Non-Uniform Scaling

- To scale a vector u by a non-uniform amount (a, b, c):

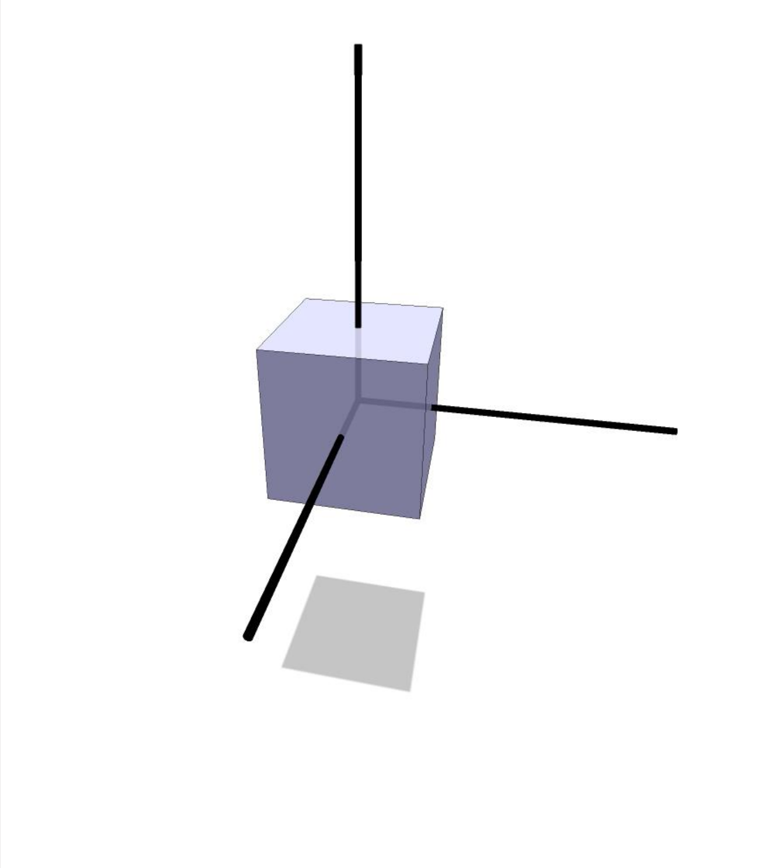
$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} au_1 \\ bu_2 \\ cu_3 \end{bmatrix}$$

- The above works only if scaling is axis-aligned. What if it isn't?
- Idea:
  - Rotate to a new axis R
  - Perform axis-aligned scaling D
  - Rotate back to original axis R^T

$$A \coloneqq R^T D R$$

  - Resulting transform A is a symmetric matrix

- **Q:** Do all symmetric matrices represent non-uniform scaling?



Spectral Theorem

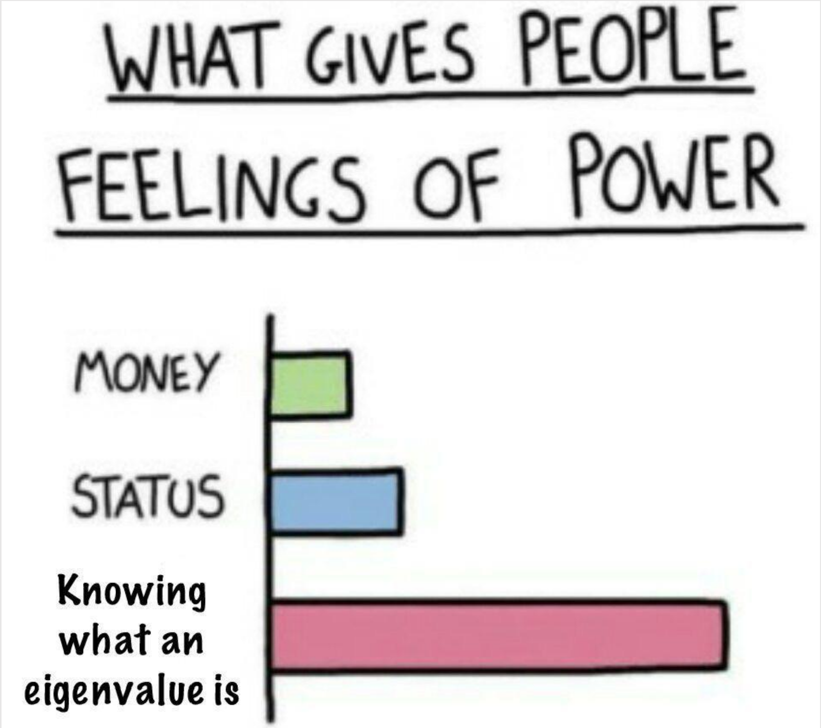
- **Spectral theorem** says a symmetric matrix $A = A^T$ has:
  - Orthonormal eigenvectors $e_1, \ldots, e_n \in \mathbb{R}^n$
  - Real eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$

- Eigenvalues represent the diagonals of the scalar transform
- Eigenvectors are axis which we are scaling about
  - Can be represented as a rotation transform

$$R = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- Can write the relationship as $AR = RD$
  - Equivalently, $A = RDR^{\mathsf{T}}$

- Hence, every symmetric matrix performs a non-uniform scaling along some set of orthogonal axes



Shear

- A shear displaces each point x in a direction u according to its distance along a fixed vector v:

$$f_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = \mathbf{x} + \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{u}$$

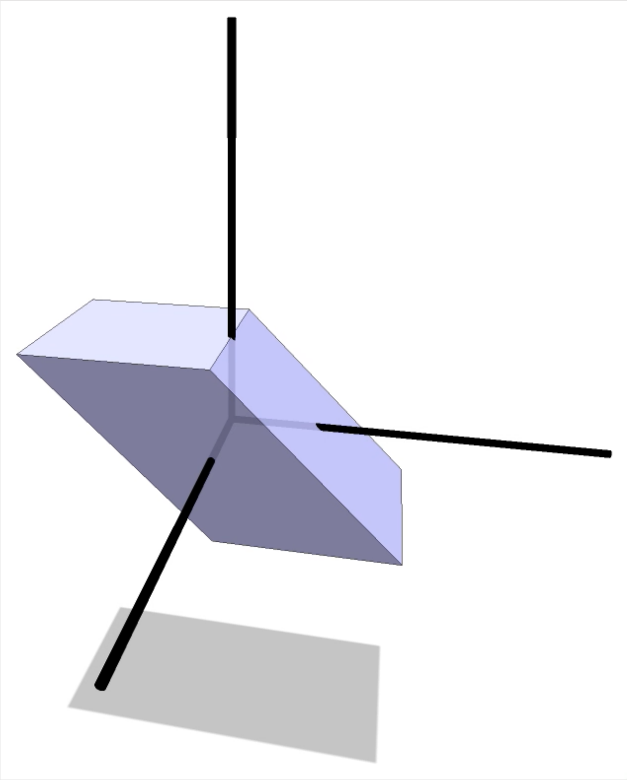
- Still a linear transformation—can be rewritten as:

$$A_{\mathbf{u},\mathbf{v}} = I + \mathbf{u}\mathbf{v}^{\mathsf{T}}$$

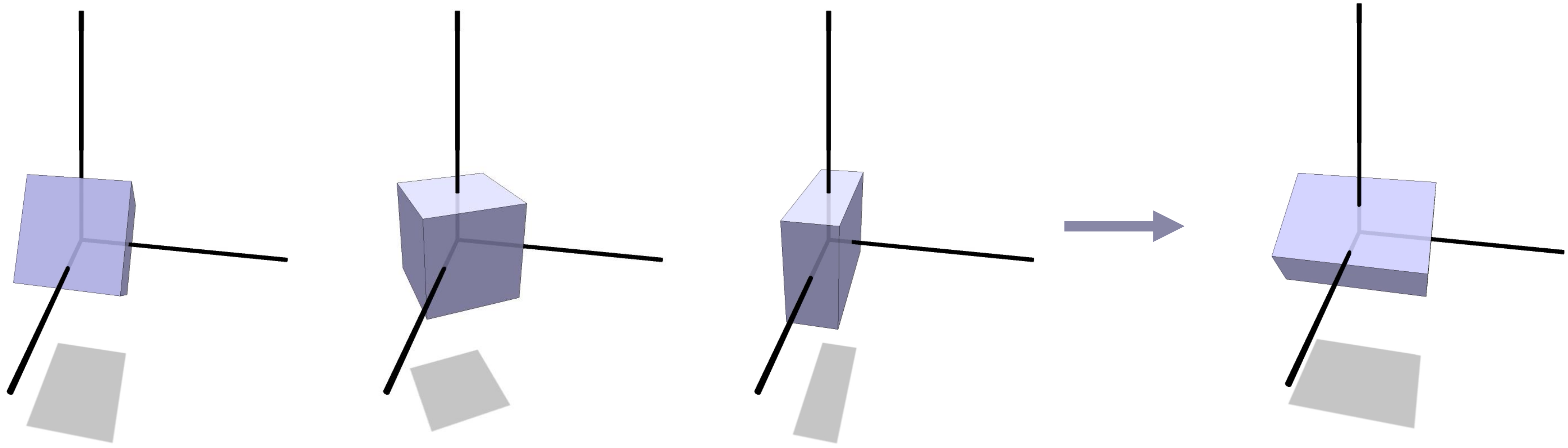
- Example:

$$\mathbf{u} = (\cos(t), 0,0)$$
$$\mathbf{v} = (0,1,0)$$

$$A_{\mathbf{u},\mathbf{v}} = \begin{bmatrix} 1 & \cos(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Composing Transforms

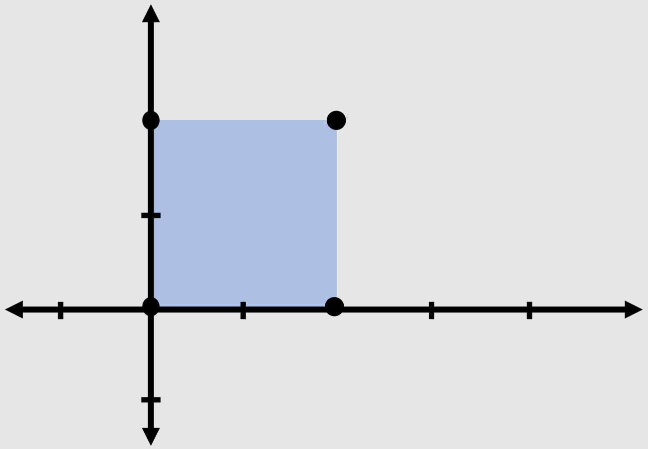


$R_x(t)$ $R_y(t)$ $S(t)$ $A(t) = R_x(t)R_y(t)S(t)$

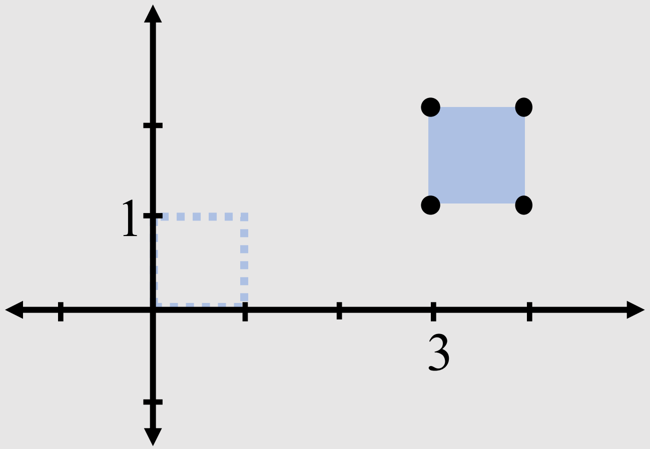
We can now build up composite transformations via matrix multiplication

Composing Transforms

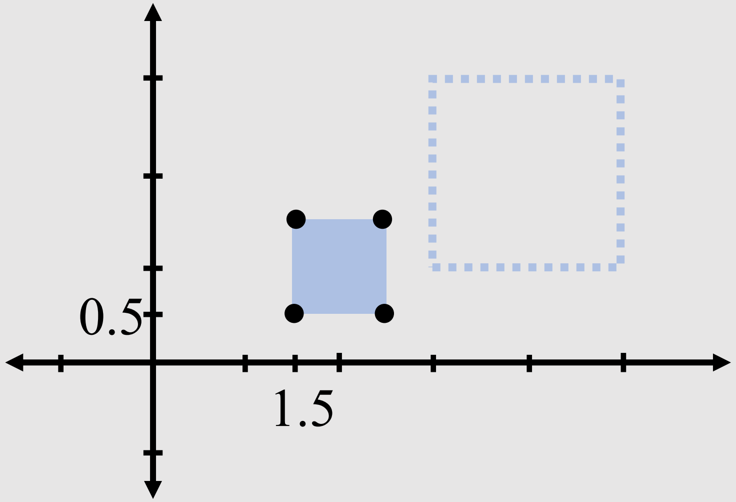
- Order matters when compositing transforms!



[original]



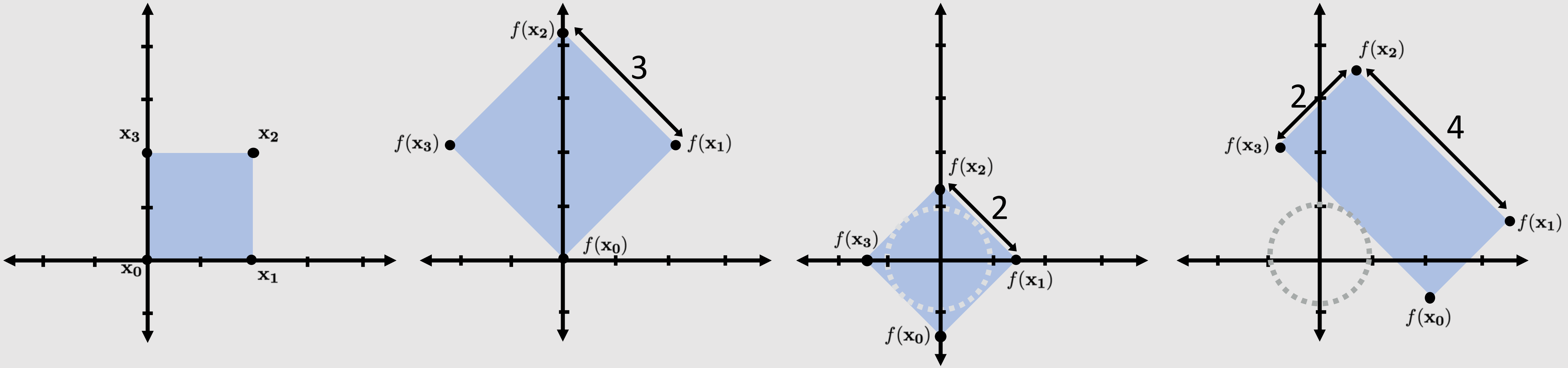
[scale by 1/2, then translate by (3,1)]



[translate by (3,1), then scale by 1/2]

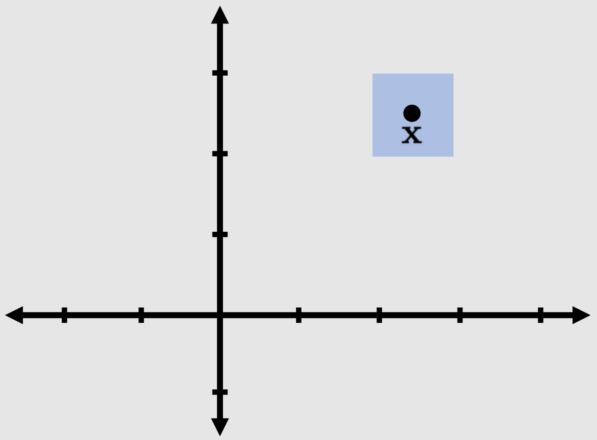
Composing Transforms

How would you perform these transformations?**

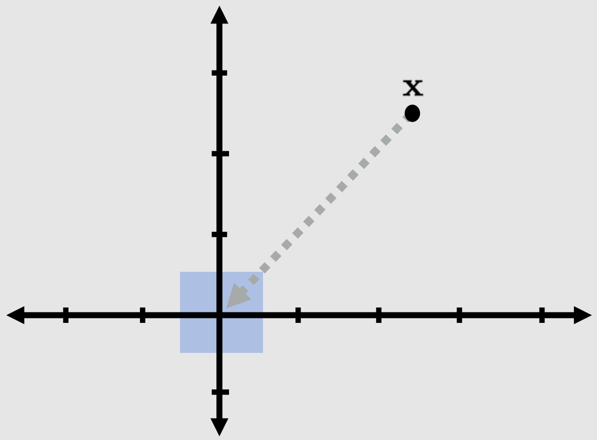


**remember there's always more than one way to do so

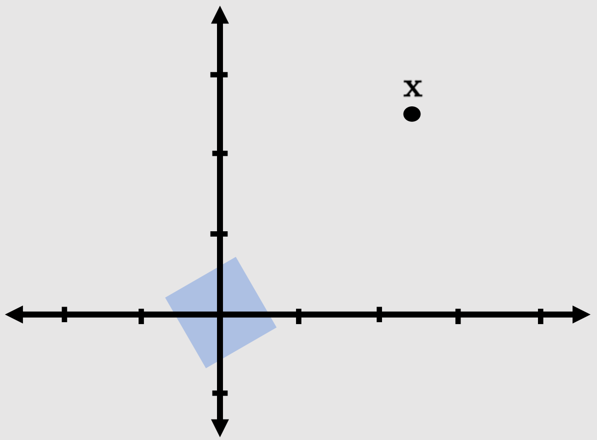
Rotating About A Point



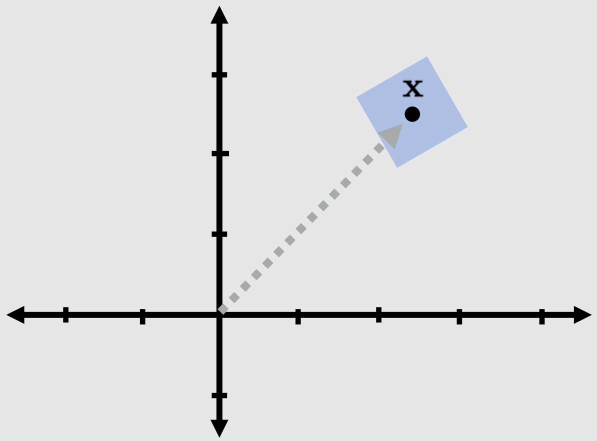
[Step 0] compute x (dist. from origin)



[Step 1] translate by -x



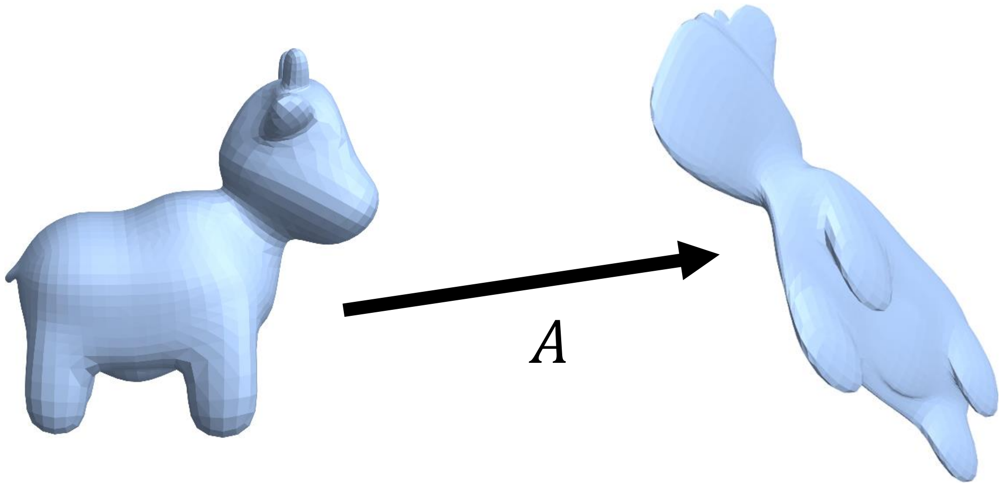
[Step 2] rotate



[Step 3] translate by x

Decomposing Transforms

- In general, no **unique** way to write a given linear transformation as a composition of basic transformations!
  - However, there are many useful decompositions:
    - **Singular value decomposition**
      - Good for signal processing
    - **LU factorization**
      - Good for solving linear systems
    - **Polar decomposition**
      - Good for spatial transformations



$$A = \begin{bmatrix} .34 & -.11 & -.89 \\ -.65 & .52 & -.70 \\ .25 & .23 & -.69 \end{bmatrix}$$

Polar & Single Value Decomposition

Polar decomposition decomposes any matrix A into orthogonal matrix Q and symmetric positive-semidefinite matrix P

rotation/reflection

nonnegative nonuniform scaling

$$A = QP$$

Since P is symmetric, can take this further via the spectral decomposition $P = VDV^T$ (V orthogonal, D diagonal):

$$A = \underbrace{QV}_{U} DV^{\mathsf{T}} = UDV^{\mathsf{T}}$$

rotation

axis-aligned scaling

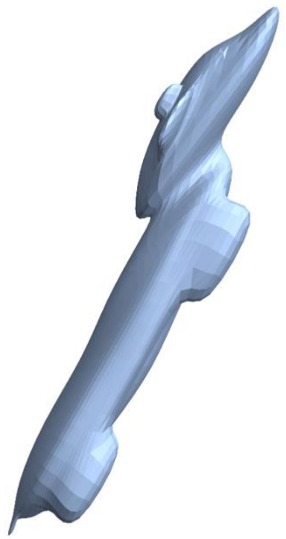
rotation

Result $\mathrm{U}DV^T$ is called the **singular value decomposition**

Interpolating Transformations [Linear]

Consider interpolating between two linear transformations
A_0, A_1 of some initial model

Idea: take a linear combination of the two matrices



$$A(t) = (1-t)A_0 + tA_1$$

$$t \in [0,1]$$

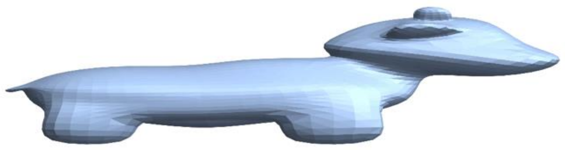
Hits the right start/endpoints… but looks awful in between!

Interpolating Transformations [Polar]

Better idea: separately interpolate components of polar decomposition

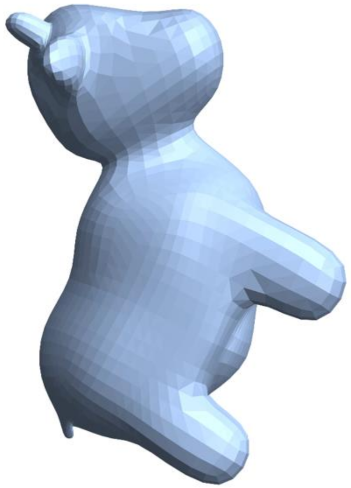
$$A_0 = Q_0 P_0$$
$$A_1 = Q_1 P_1$$

[scaling]



$$P(t) = (1-t)P_0 + tP_1$$

[rotation]



$$Q(t) = (1-t)Q_0 + tQ_1$$

[composite]

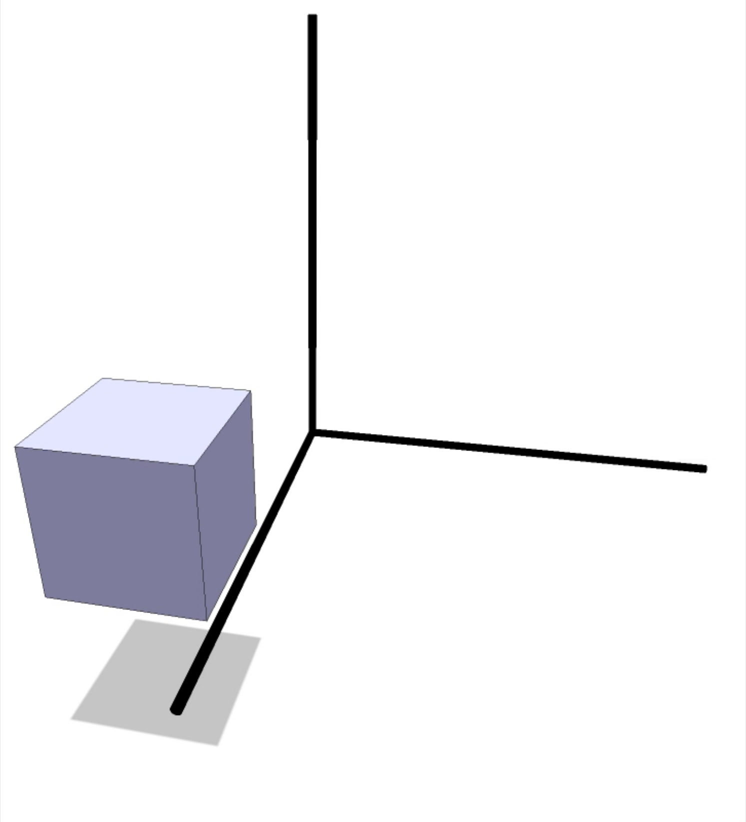
$$A(t) = Q(t)P(t)$$

Translation

- So far we've ignored a basic transformation—translations
  - A translation simply adds an offset $\mathbf{u}$ to the given point $\mathbf{x}$

$$f_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} + \mathbf{u}$$

- Is this translation linear?
  - (certainly seems to move across a line…)



[additivity]

$$f_{\mathbf{u}}(\mathbf{x} + \mathbf{y}) = \mathbf{x} + \mathbf{y} + \mathbf{u}$$
$$f_{\mathbf{u}}(\mathbf{x}) + f_{\mathbf{u}}(\mathbf{y}) = \mathbf{x} + \mathbf{y} + 2\mathbf{u}$$

[homogeneity]

$$f_{\mathbf{u}}(a\mathbf{x}) = a\mathbf{x} + \mathbf{u}$$
$$a f_{\mathbf{u}}(\mathbf{x}) = a\mathbf{x} + a\mathbf{u}$$

Translations are not linear!

Maybe translations turn linear when we go into the 4^{th} dimension…

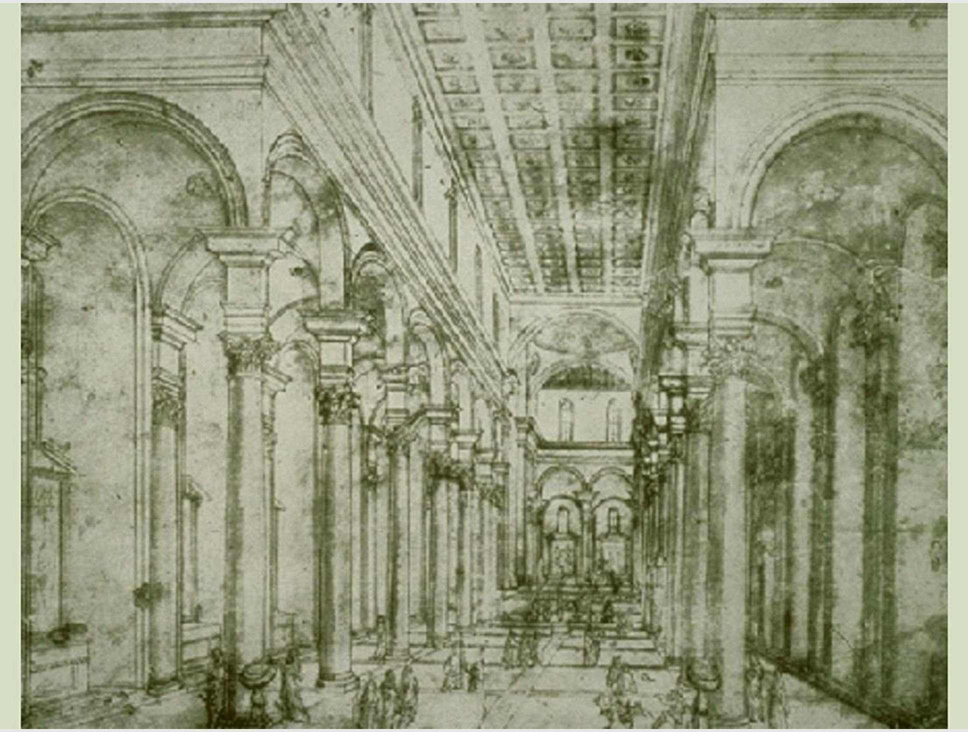
- ~~The Rasterization Pipeline~~
- ~~Transformations~~
- Homogeneous Coordinates
- 3D Rotations

Homogeneous Coordinates

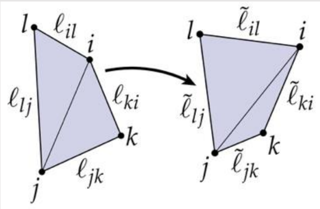
- Came from efforts to study perspective

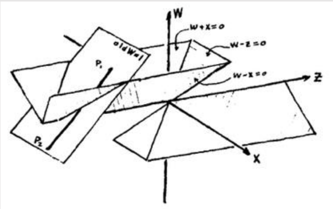
- Introduced by Möbius as a natural way of assigning coordinates to lines

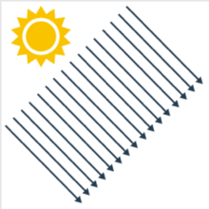
- Show up naturally in a surprising large number of places in computer graphics:
  - 3D transformations
  - Perspective projection
  - Quadric error simplification
  - Premultiplied alpha
  - Shadow mapping
  - Projective texture mapping
  - Discrete conformal geometry
  - Hyperbolic geometry
  - Clipping
  - Directional lights
  - …



Church of Santo Spirito (1428) Filippo Brunelleschi

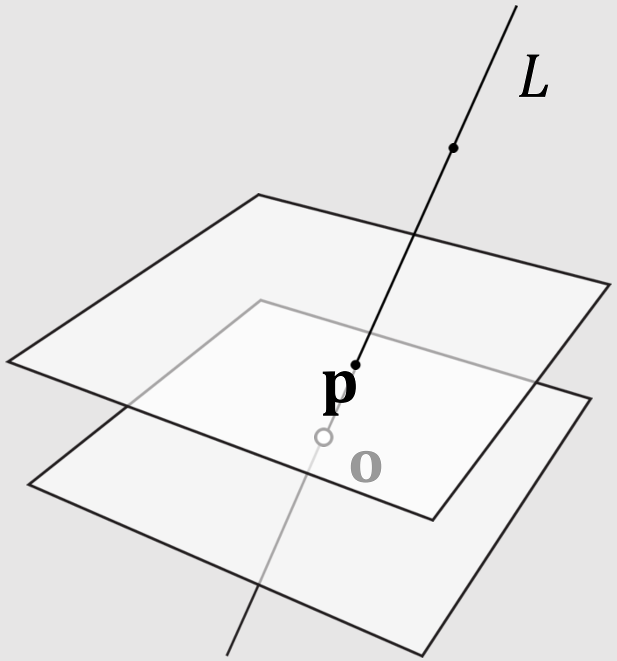


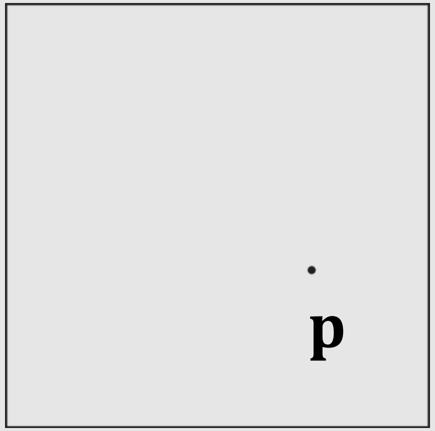




Homogeneous Coordinates in 2D

- Consider any 2D plane that does not pass through the origin o in 3D
  - Every line through the origin in 3D corresponds to a point in the 2D plane
  - Just find the point p where the line L pierces the plane
- Consider a point $p' = (x, y)$, and the plane $z = 1$ in 3D
  - Any three numbers $p = (a, b, c)$ such that $\left(\frac{a}{c}, \frac{b}{c}\right) = (x, y)$ are homogeneous coordinates for p
    - Example: $(x, y, 1)$
    - In general: (cx, cy, c) for $c \neq 0$
      - The c is commonly referred to as the homogeneous coordinate
- Great, but how does this help us with transforms?





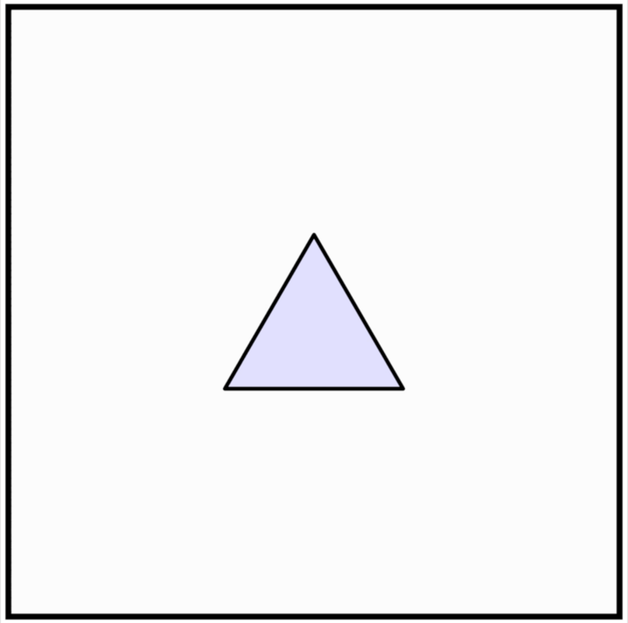
Translation in Homogeneous Coordinates

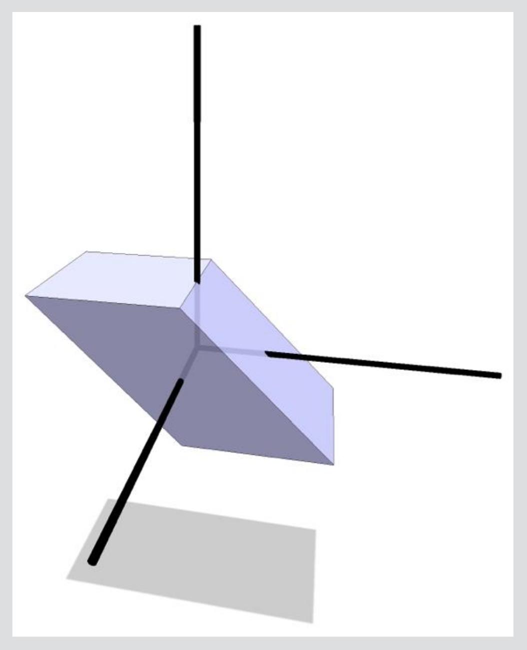
- A 2D translation is similar to a 3D shear
  - Moving a slice up/down the shear moves the shape
- Recall shear is written as:

$$f_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = \mathbf{x} + \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{u}$$

$$f_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = \left(I + \mathbf{u}\mathbf{v}^{\mathsf{T}}\right)\mathbf{x}$$

- In our case, $v = (0, 0, 1)$, so**

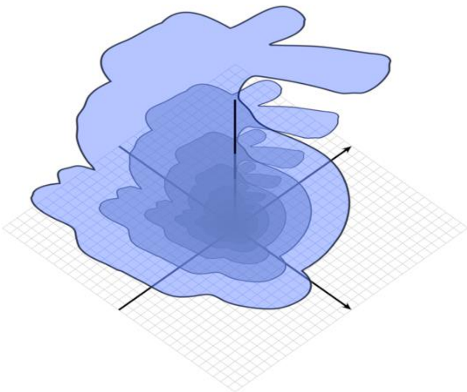




$$\begin{bmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cp_1 \\ cp_2 \\ c \end{bmatrix} = \begin{bmatrix} c(p_1 + u_1) \\ c(p_2 + u_2) \\ c \end{bmatrix} \xRightarrow{1/c} \begin{bmatrix} p_1 + u_1 \\ p_2 + u_2 \end{bmatrix}$$

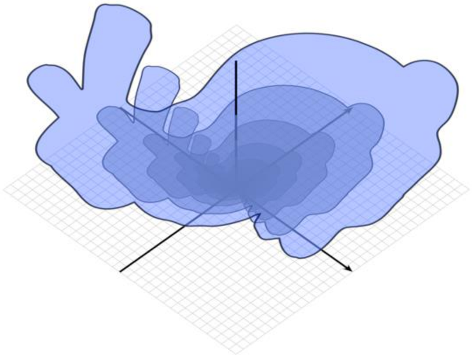
**most often in this class we will also use c = 1

2D Transforms in Homogeneous Coordinate



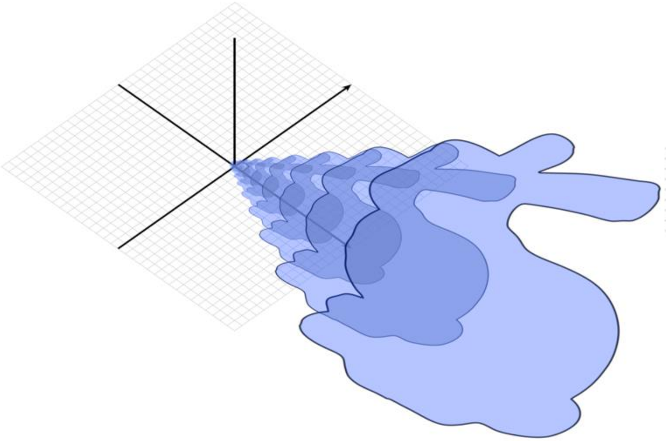
[original]

Original shape in 2D can be viewed as many copies along the z-axis



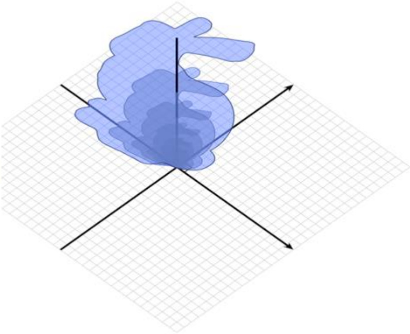
[2D rotation]

Rotate around the z-axis



[2D translate]

Shear in direction of translation



[2D scale]

Scale x-axis and y-axis, preserve z-axis

Q: What about 3D homogeneous coordinates?

3D Transforms in Homogeneous Coordinate

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

[point in 3D]

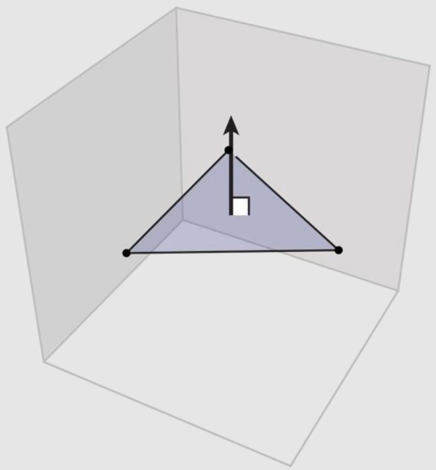
Matrix representations of 3D linear transformations just get an additional identity row/column:

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & s & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

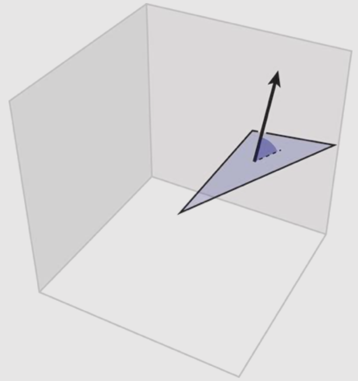
[rotate around y by θ] [shear by z in (s,t) direction] [scale by a,b,c] [translate by (u,v,w)]

Points vs. Vectors

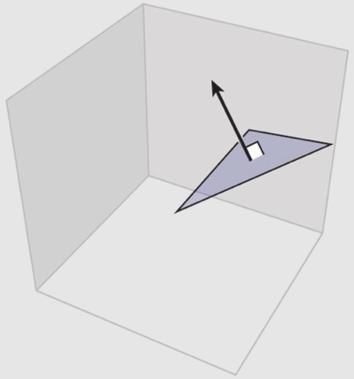
- Homogeneous coordinates should be used differently for points and vectors:
  - Triangle vertices are "points" and should be translated and rotated
    - But if we do the same for the normal, it no longer becomes a normal
  - **Idea:** normal is a "vector" and should just rotate!**
    - Set homogeneous coordinate to 0



$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & u \\ 0 & 1 & 0 & v \\ -\sin\theta & 0 & \cos\theta & w \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ 1 \end{bmatrix}$$



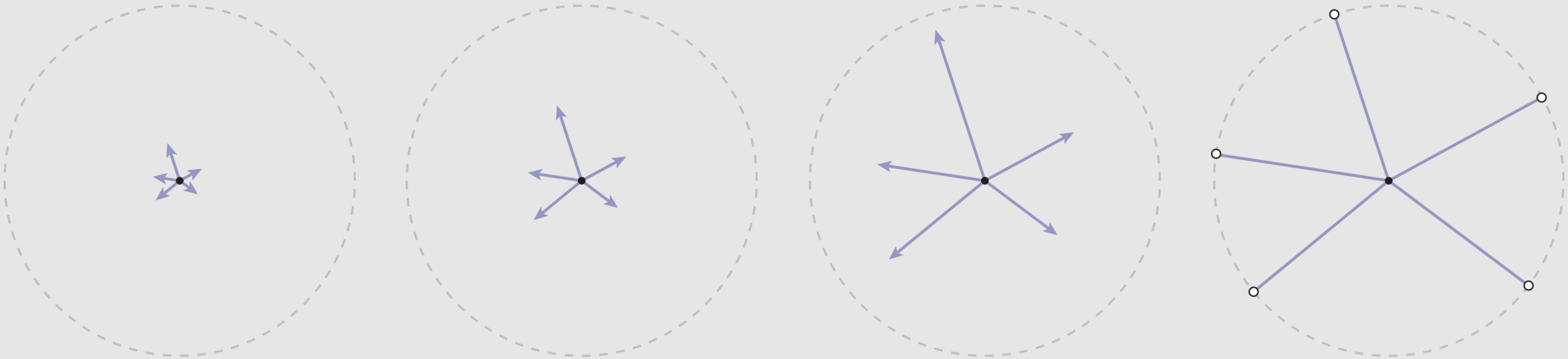
$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & u \\ 0 & 1 & 0 & v \\ -\sin\theta & 0 & \cos\theta & w \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ 0 \end{bmatrix}$$



**translating or scaling a triangle should never change the normal

Points vs. Vectors in Homogeneous Coordinates

- In general:
  - A point has a nonzero homogeneous coordinate (c = 1)
  - A vector has a zero homogeneous coordinate (c = 0)
- But wait… what division by c mean when it's equal to zero?
- Well consider what happens as c approaches 0…



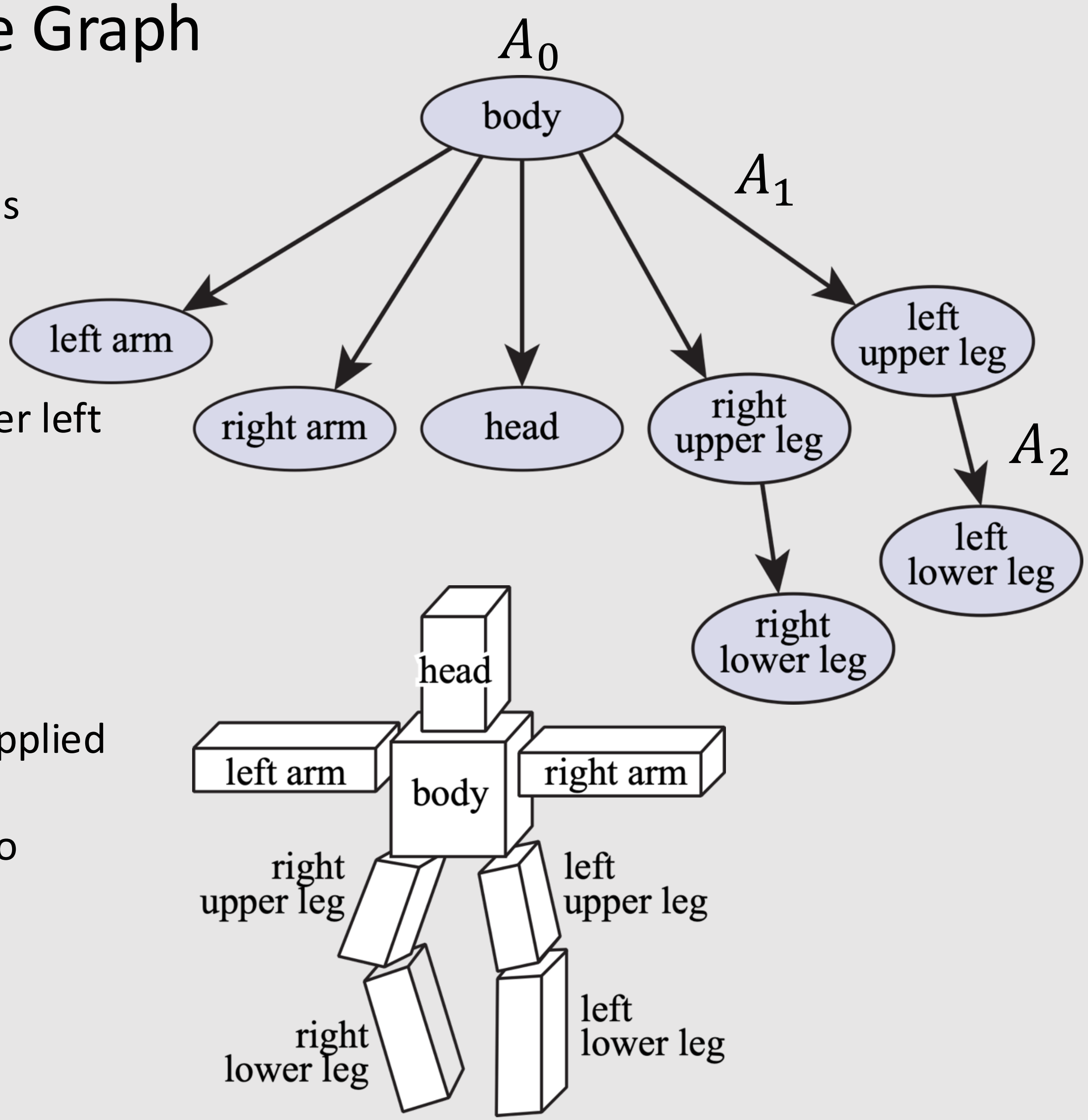
$(x, y)/1$ $(x, y)/0.5$ $(x, y)/0.25$ $(x, y)/0.001$

- Can think of vectors as "points at infinity" (sometimes called "ideal points")
  - **But don't actually go dividing by zero…**

Where can we use transforms?

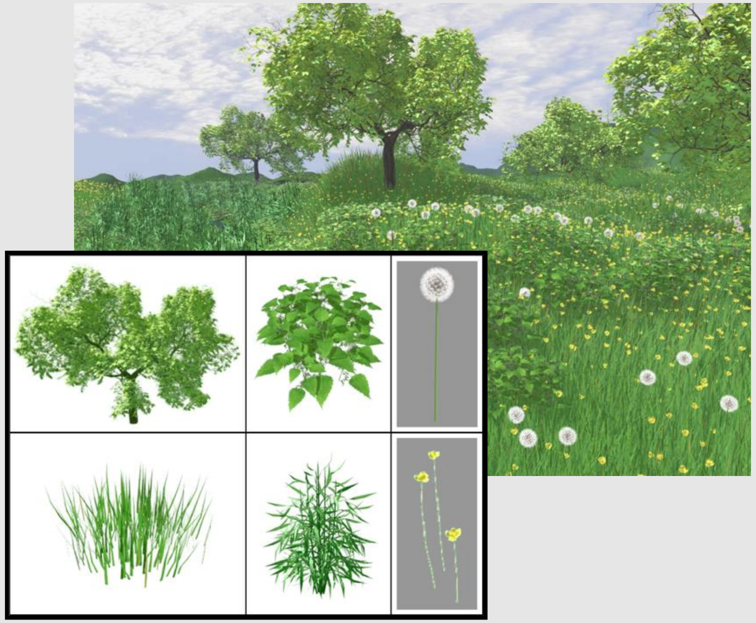
Scene Graph

- Suppose we want to build a skeleton out of cubes
  - **Idea:** transform cubes in world space
    - Store transform of each cube
- **Problem:** If we rotate the left upper leg, the lower left leg won't track with it
  - **Better Idea:** store a hierarchy of transforms
    - Known as a **scene graph**
    - Each edge (+root) stores a linear transformation
    - Composition of transformations gets applied to nodes
      - Keep transformations on a stack to reduce redundant multiplication
- **Lower left leg transform:** $A_0 A_1 A_2$

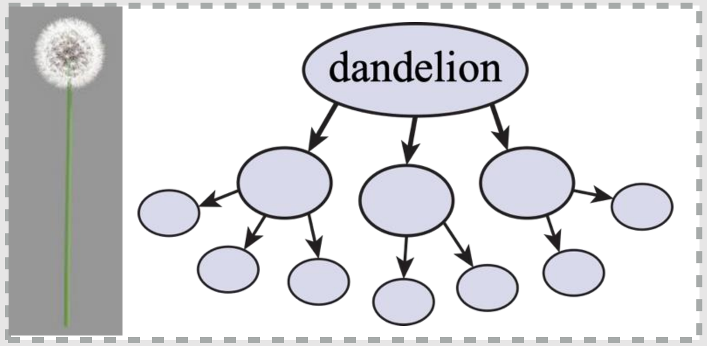


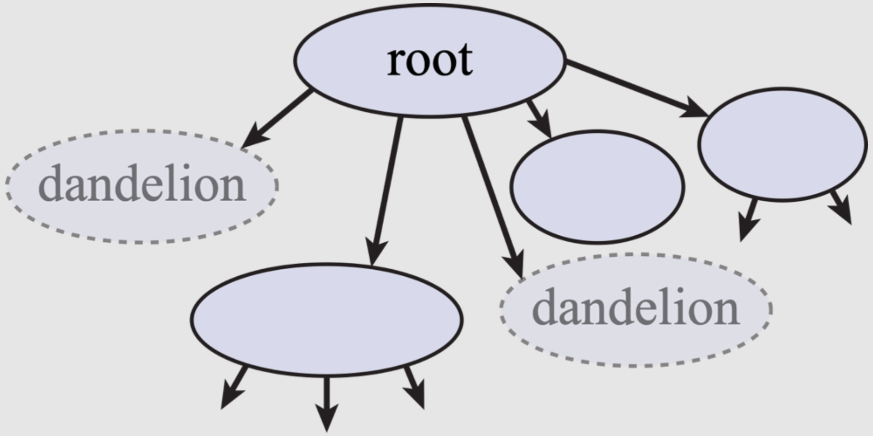
Instancing

- What if we want many copies of the same object in a scene?
  - Rather than have many copies of the geometry, scene graph, we can just put a “pointer” node in our scene graph
    - Saves a reference to a shared geometry
    - Specify a transform for each reference
      - **Careful!** Modifying the geometry will modify all references to it



Realistic modeling and rendering of plant ecosystems (1998) Deussen et al



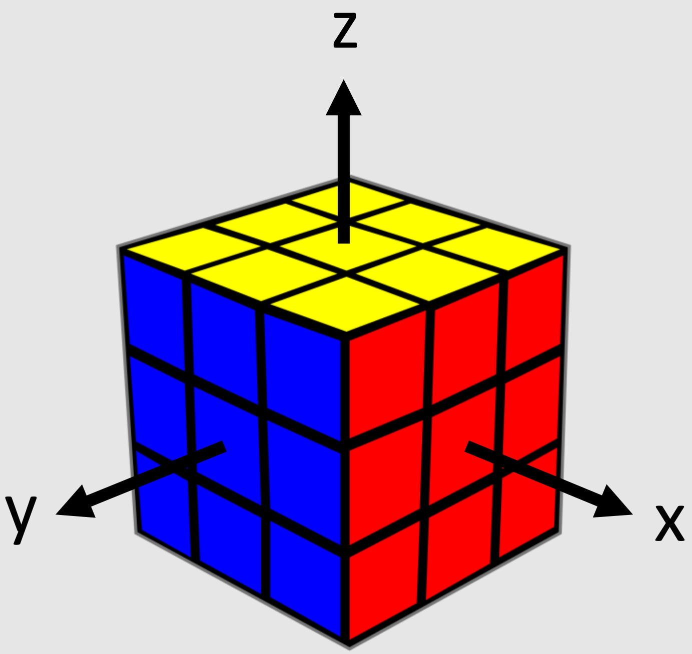


- ~~The Rasterization Pipeline~~
- ~~Transformations~~
- ~~Homogeneous Coordinates~~
- 3D Rotations

3D Rotations

- Rotating in 2D is the same as rotating around the z-axis
- **Idea:** independently rotate around each (x,y,z)-axis for 3D rotations

- **Problem:** order of rotation matters!
  - Rotate a Rubik's cube 90deg around the y-axis and 90deg around the z-axis
  - Rotate a Rubik's cube 90deg around the z-axis and 90deg around the y-axis
    - They will not be the same!
  - Order of rotation must be specified



3D Rotations in Matrix Form

Idea: independently rotate around each (x,y,z)-axis for 3D rotations:

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix} \quad R_y = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \quad R_z = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Combining the matrices:

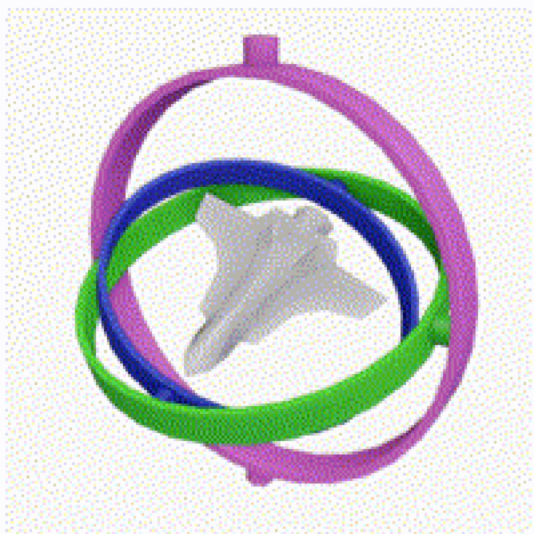
$$R_xR_yR_z = \begin{bmatrix} \cos\theta_y\cos\theta_z & -\cos\theta_y\sin\theta_z & \sin\theta_y \\ \cos\theta_z\sin\theta_x\sin\theta_y + \cos\theta_x\sin\theta_z & \cos\theta_x\cos\theta_z - \sin\theta_x\sin\theta_y\sin\theta_z & -\cos\theta_y\sin\theta_x \\ -\cos\theta_x\cos\theta_z\sin\theta_y + \sin\theta_x\sin\theta_z & \cos\theta_z\sin\theta_x + \cos\theta_x\sin\theta_y\sin\theta_z & \cos\theta_x\cos\theta_y \end{bmatrix}$$

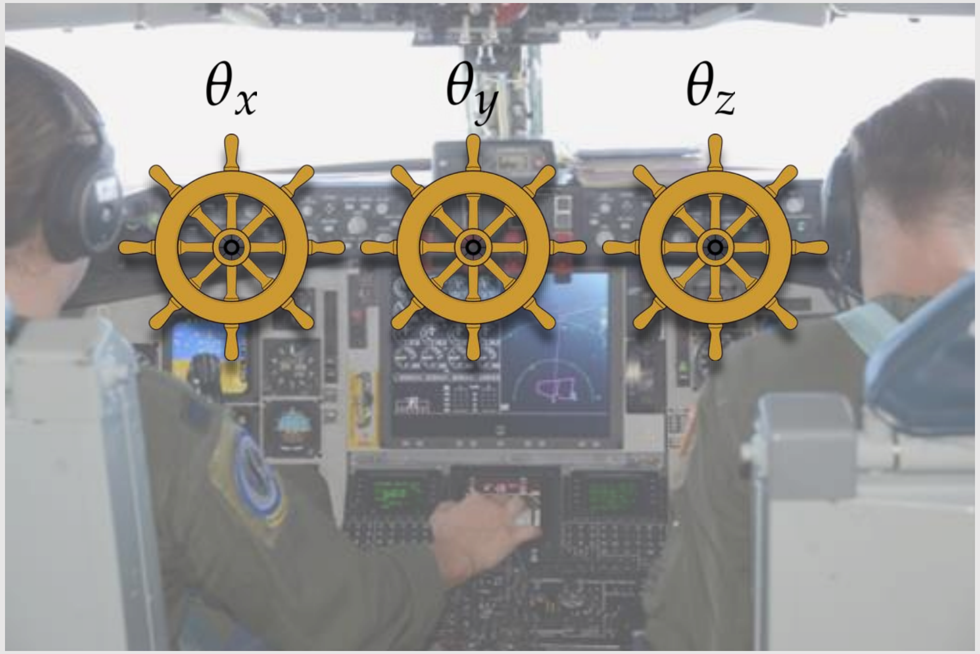
Consider the special case $\theta_y = \pi/2$ (so, $\cos\theta_y = 0$, $\sin\theta_y = 1$):

$$\Longrightarrow \begin{bmatrix} 0 & 0 & 1 \\ \cos\theta_z\sin\theta_x + \cos\theta_x\sin\theta_z & \cos\theta_x\cos\theta_z - \sin\theta_x\sin\theta_z & 0 \\ -\cos\theta_x\cos\theta_z + \sin\theta_x\sin\theta_z & \cos\theta_z\sin\theta_x + \cos\theta_x\sin\theta_z & 0 \end{bmatrix}$$

Gimbal Lock

- **No matter how we adjust θx, θz, can only rotate in one plane!**
- We are now “locked” into a single axis of rotation
  - Not a great design for airplane controls!





$$\Longrightarrow \begin{bmatrix} 0 & 0 & 1 \\ \cos\theta_z \sin\theta_x + \cos\theta_x \sin\theta_z & \cos\theta_x \cos\theta_z - \sin\theta_x \sin\theta_z & 0 \\ -\cos\theta_x \cos\theta_z + \sin\theta_x \sin\theta_z & \cos\theta_z \sin\theta_x + \cos\theta_x \sin\theta_z & 0 \end{bmatrix}$$

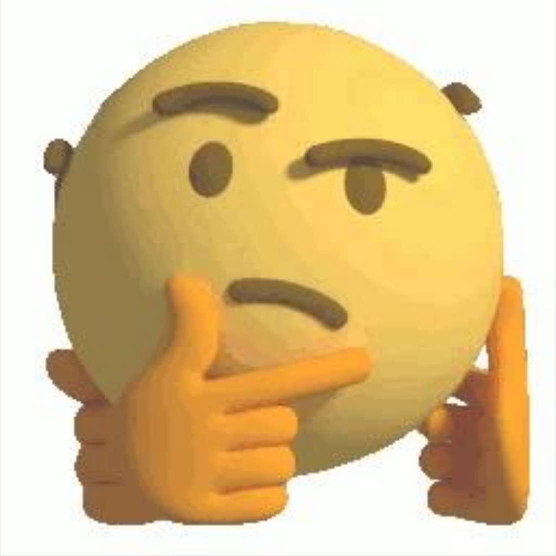
Rotation From Axis/Angle

Alternatively, there is a general expression for a matrix that performs a rotation around a given axis u by a given angle θ:

$$\begin{bmatrix} \cos\theta + u_x^2(1-\cos\theta) & u_x u_y(1-\cos\theta) - u_z\sin\theta & u_x u_z(1-\cos\theta) + u_y\sin\theta \\ u_y u_x(1-\cos\theta) + u_z\sin\theta & \cos\theta + u_y^2(1-\cos\theta) & u_y u_z(1-\cos\theta) - u_x\sin\theta \\ u_z u_x(1-\cos\theta) - u_y\sin\theta & u_z u_y(1-\cos\theta) + u_x\sin\theta & \cos\theta + u_z^2(1-\cos\theta) \end{bmatrix}$$

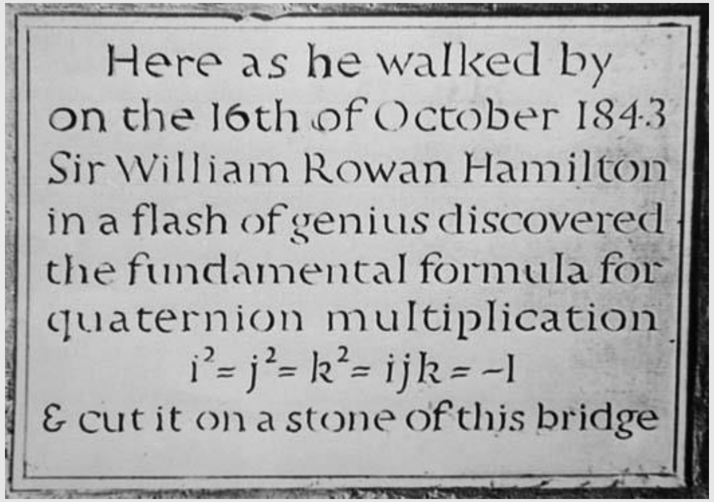
Just memorize this matrix! :)

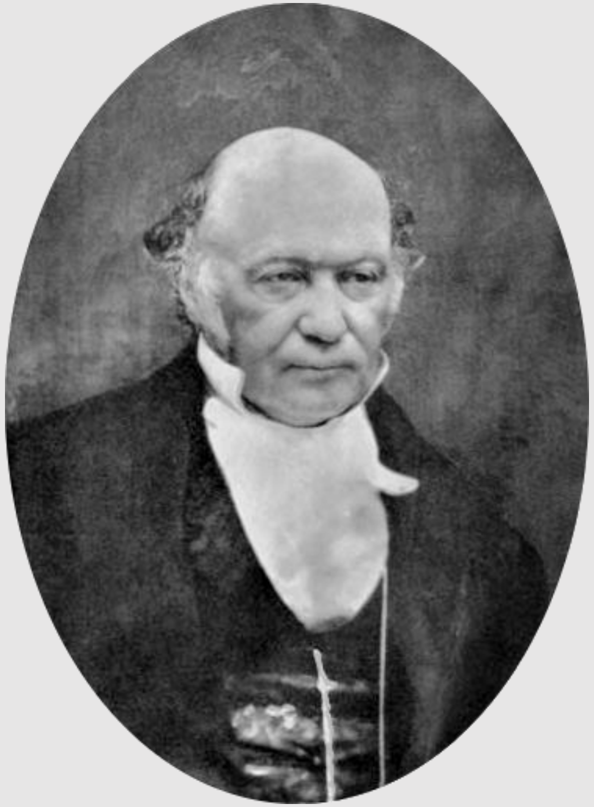
Is there a better way to perform 3D rotations?



Bridging The Rotation Gap

- Hamilton wanted to make a 3D equivalent for complex numbers
  - One day, when crossing a bridge, he realized he needed 4 (not 3) coordinates to describe 3D complex number space
    - 1 real and 3 complex components
  - He carved his findings onto a bridge (still there in Dublin)
  - Later known as quaternions





William Rowan Hamilton
[1805 – 1865]

Quaternions For Math People

- 4 coordinates (1 real, 3 complex) comprise coordinates.
  - H is known as the ‘Hamilton Space’

$$\mathbb{H} := \text{span}(\{1, \imath, \jmath, k\})$$

$$q = a + b\imath + c\jmath + dk \in \mathbb{H}$$

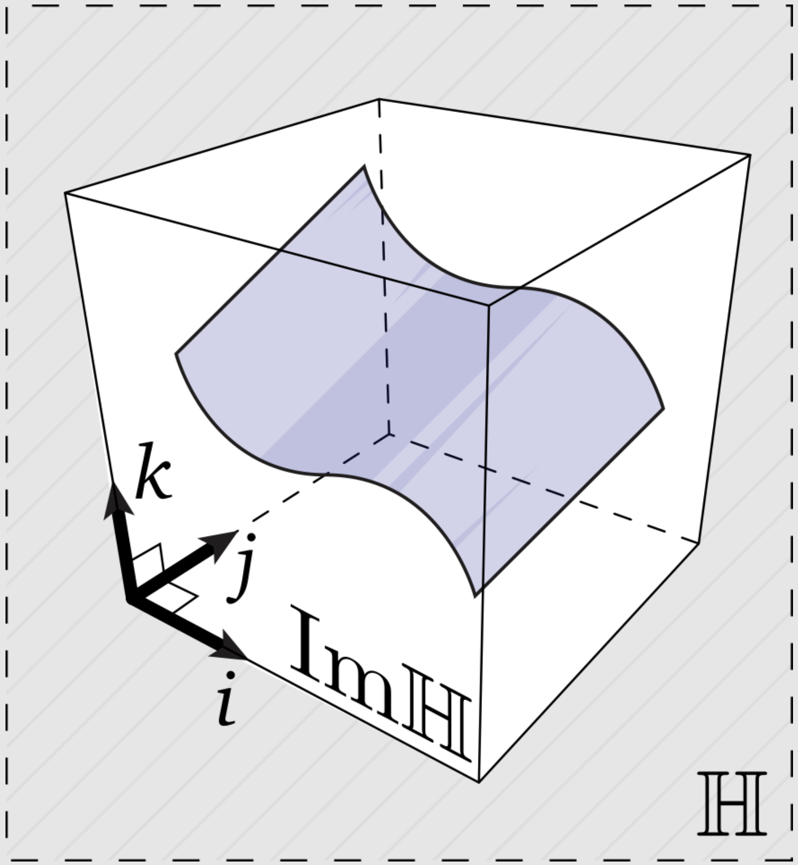
- Quaternion product determined by:

$$\imath^2 = \jmath^2 = k^2 = \imath\jmath k = -1$$

- **Warning:** product no longer commutes!

$$\text{For } q, p \in \mathbb{H}, \quad qp \neq pq$$

  - With 3D rotations, order matters.



Quaternions For Non-Math People

- Recall axis-angle rotations
  - Represent an axis with 3 coordinates (i, j, k)
  - Represent an angle by some scalar a

$$q = a + b\imath + c\jmath + dk \in \mathbb{H}$$

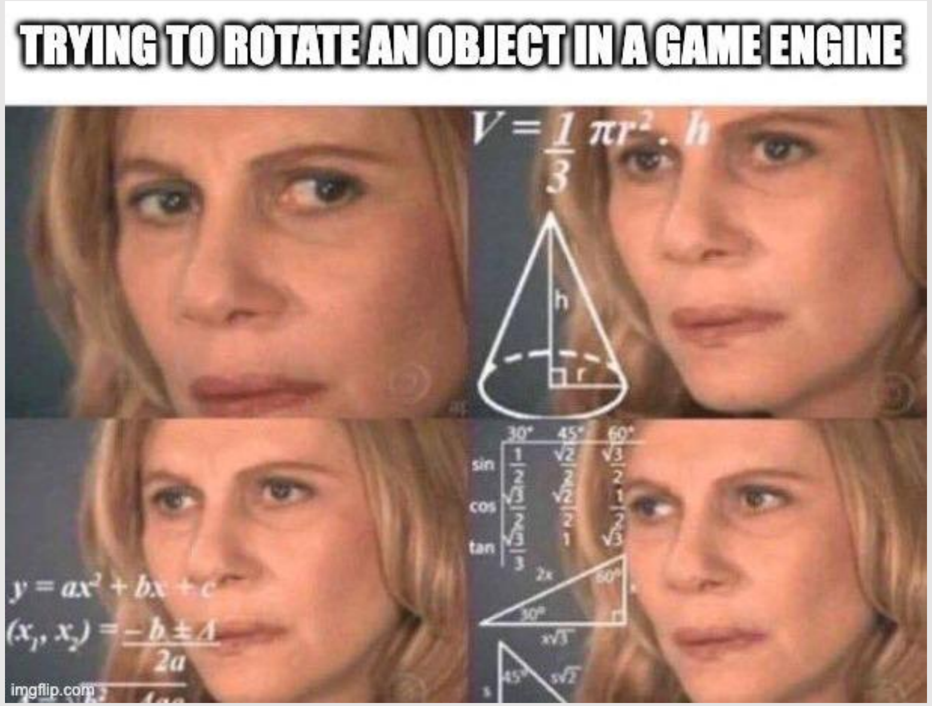
- Just like how we multiply rotation matrices together, we can also multiply complex components. If we represent:
  - i as a 90deg rotation about x-axis
  - j as a 90deg rotation about y-axis
  - k as a 90deg rotation about z-axis

$$\imath^2 = \jmath^2 = k^2 = \imath\jmath k = -1$$

  - Then two 90deg rotations about the same axis will produce the inverted image, the same as scaling by -1
  - This can also be rewritten as:

$$ij = k$$

    - A 90deg x-axis rotation and a 90deg y-axis rotation is the same as a 90deg z-axis rotation
    - Can be rewritten in any other way



Multiplying Quaternions

Given two quaternions:

$$q = a_1 + b_1 \imath + c_1 \jmath + d_1 k$$
$$p = a_2 + b_2 \imath + c_2 \jmath + d_2 k$$

Can express their product as:

$$\begin{aligned} qp = {} & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \\ & +(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\imath \\ & +(a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)\jmath \\ & +(a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)k \end{aligned}$$

recall

$$\imath^2 = \jmath^2 = k^2 = \imath\jmath k = -1$$

The result still looks like a quaternion
But there's a better way to multiply…

Multiplying Quaternions

Recall quaternions can be thought of as an axis and angle:

$$(x, y, z) \mapsto 0 + x\imath + y\jmath + zk$$

$$(\underset{\mathbb{R}}{\underset{\cap}{\text{scalar},}}\ \underset{\mathbb{R}^3}{\underset{\cap}{\text{vector}}}) \in \mathbb{H}$$

Can express their product as:

$$(a, \mathbf{u})(b, \mathbf{v}) = (ab - \mathbf{u} \cdot \mathbf{v}, a\mathbf{v} + b\mathbf{u} + \mathbf{u} \times \mathbf{v})$$

If the scalar components are 0, we get:

$$\mathbf{uv} = \mathbf{u} \times \mathbf{v} - \mathbf{u} \cdot \mathbf{v}$$

Rotating With Quaternions

- **Goal:** rotate x by angle θ around axis $u = (x, y, z)$:
  - Make x imaginary, and build q based on u and θ
    - **Note:** components of q must be normalized!

$$x \in \mathrm{Im}(\mathbb{H})$$

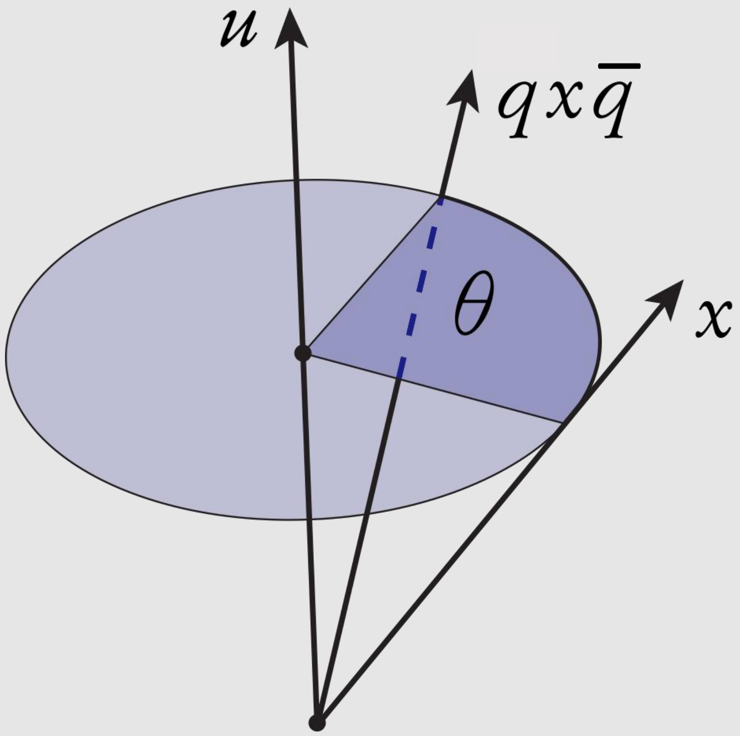
$$q \in \mathbb{H}, \quad |q|^2 = 1$$

$$q = \cos(\theta/2) + \sin(\theta/2)u$$

- q now looks like:

$$q = a + b\imath + c\jmath + dk \in \mathbb{H}$$

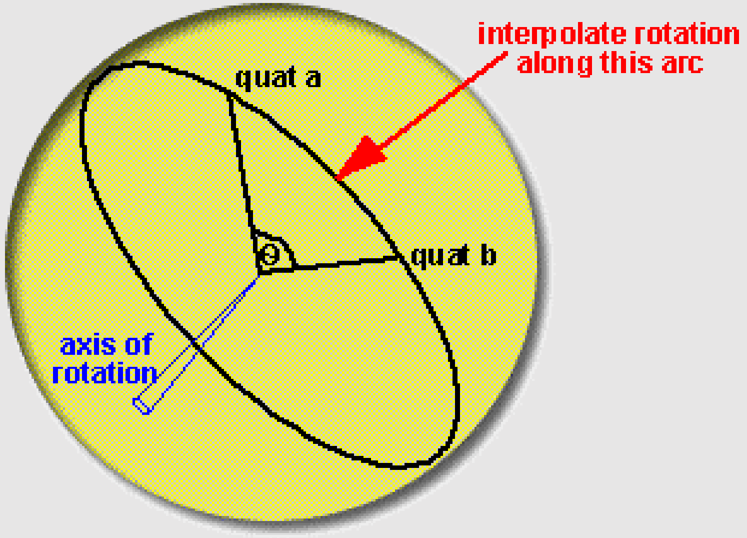
- $\bar{q}$ is q with every complex component negative
- Now just compute $qx\bar{q}$ to get final rotation



Interpolating With Quaternions

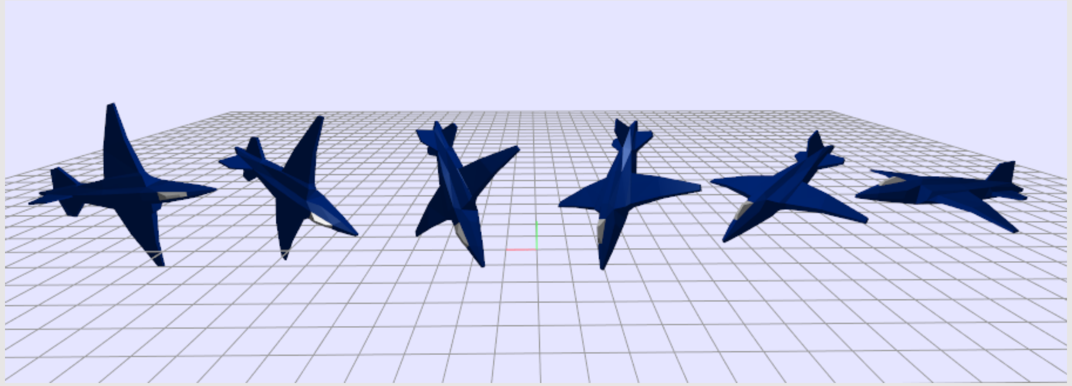
- Interpolating Euler angles can yield strange-looking paths, non-uniform rotation speed, etc.
  - Simple solution w/ quaternions: “SLERP” (spherical linear interpolation):

$$\mathrm{Slerp}(q_0, q_1, t) = q_0(q_0^{-1}q_1)^t, \quad t \in [0,1]$$



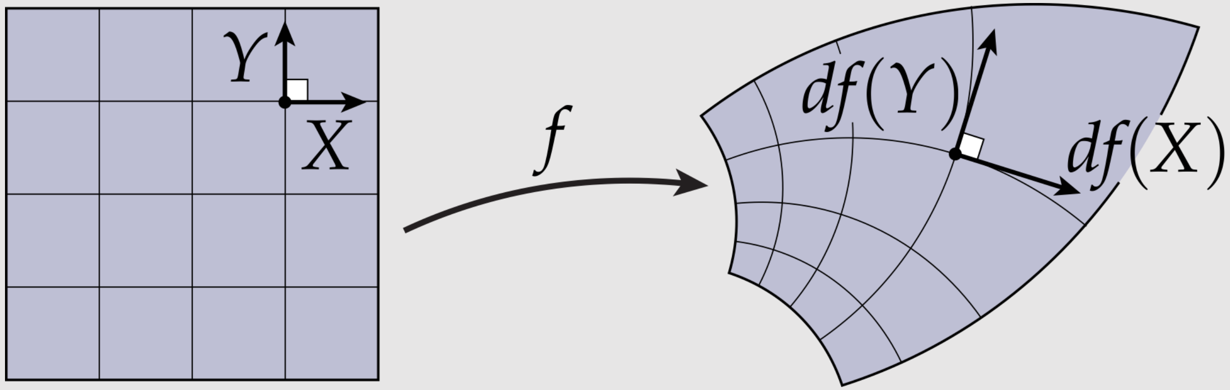
Animating Rotation with Quaternion Curves (1985) Shoemake

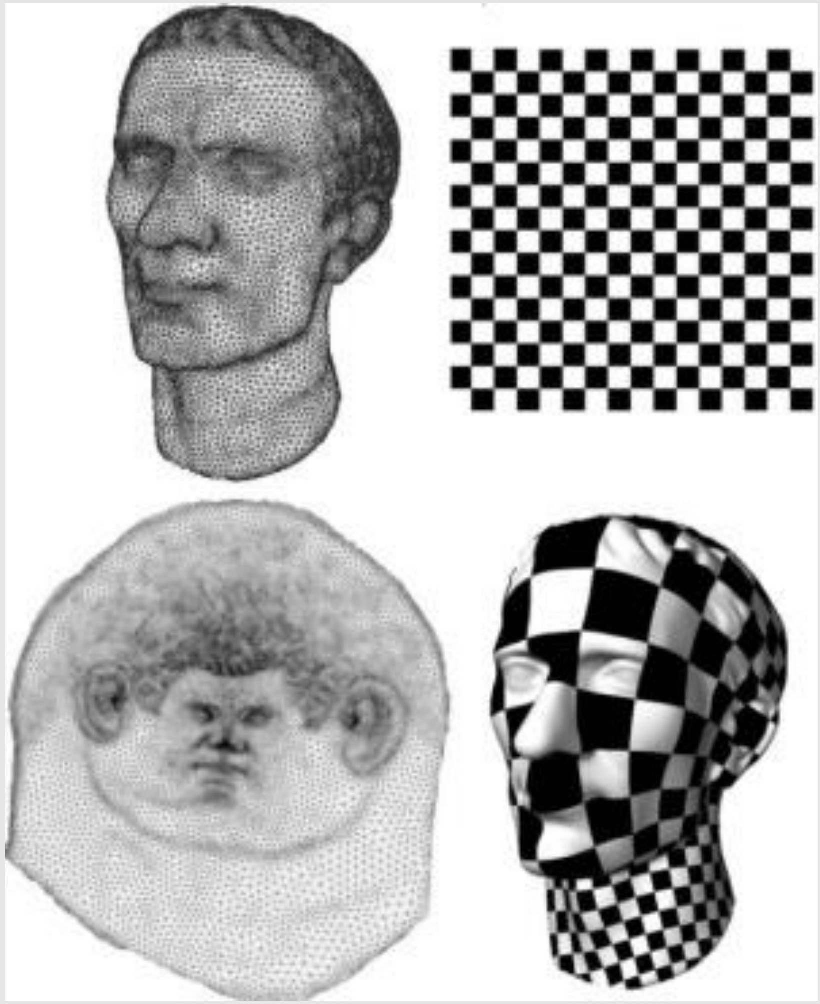
Fifa ‘15 (2014) Electronic Arts



Texture Mapping With Quaternions

- Quaternions can be used to generate texture maps coordinates
  - Complex numbers are natural language for angle-preserving (“conformal”) maps





Spatial Transformation Summary

[linear transformations]

- scaling
- rotation
- reflection
- shear

[nonlinear transformations]

- translation
- perspective projection

next lecture

Maxwell the cat (2022) Gary’s Mod

- Compose basic transformations to get more interesting ones
  - Always reduces to a single 4x4 matrix (in homogeneous coordinates)
  - Order of composition matters!
- Homogeneous coordinates can turn nonlinear transformations linear
- Many ways to decompose a given transformation (polar, SVD, …)
- Use scene graph to organize transformations
- Use instancing to eliminate redundancy
- Quaternions help avoid troubles with Euler rotations in 3D (Gimbal Lock, Interpolation inconsistencies)