The Bidirectional Reflectance Distribution Function

- BRDFs
- Materials
- Environment Lighting

Review: The Rendering Equation

$$L_o(\mathbf{p},\omega_o) \;=\; L_e(\mathbf{p},\omega_o) + \int_{\mathcal{H}^2} f_r(\mathbf{p},\omega_i \to \omega_o) L_i(\mathbf{p},\omega_i) \cos\theta \, d\omega_i$$

$L_o(\mathbf{p},\omega_o)$	outgoing radiance at point $\mathbf{p}$ in outgoing direction ω_o
$L_e(\mathbf{p},\omega_o)$	emitted radiance at point $\mathbf{p}$ in outgoing direction ω_o
$f_r(\mathbf{p},\omega_i \to \omega_o)$	scattering function at point $\mathbf{p}$ from incoming direction ω_i to outgoing direction ω_o
$L_i(\mathbf{p},\omega_i)$	incoming radiance to point $\mathbf{p}$ from direction ω_i

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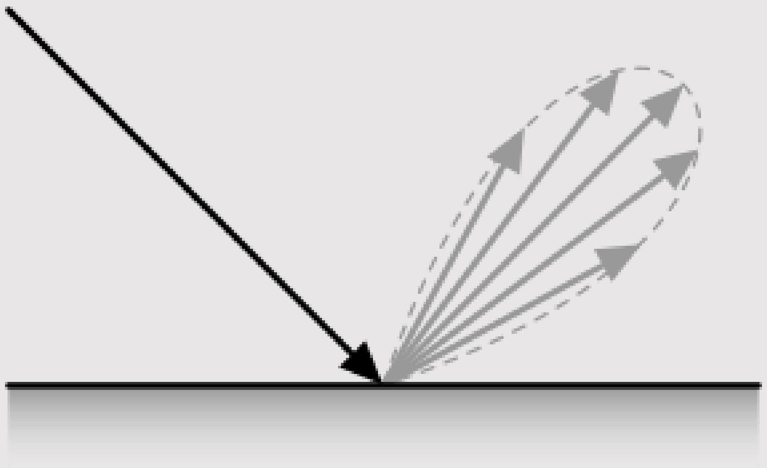
Reflectance Functions

- **Reflectance Functions** refer to how light reflects off a surface
- **Bidirectional Reflectance Distribution Function (BRDF):**
  - *Bidirectional* – a function of two directions ω_i and ω_o
  - *Reflectance* – light changing directions
  - *Distribution* – likelihood of light changing to a certain direction
  - *Function* – it's a function
- Represented as a **Probability Distribution Function (PDF)**
  - Indicating the likelihood an incident direction ω_i at point $\mathbf{p}$ will reflect to an outgoing direction ω_o



Types of Reflectance Functions

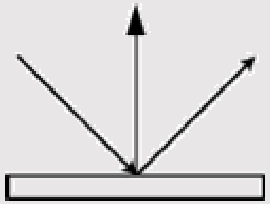
- A BRDF is a passthrough function
  - **Example:** an incoming ray ω_i at incident point **p** reflects 85% of red, 90% of green, and 50% of blue in the outgoing direction ω_o
    - Written as $f_r(\mathbf{p}, \omega_i \rightarrow \omega_o) = < 0.85, 0.90, 0.50 >$
    - Remainder of light gets absorbed
      - Conservation of energy

- Multiply the BRDF function by the incident radiance to get the outgoing radiance:

$$f_r(\mathbf{p}, \omega_i \rightarrow \omega_o) L_i(\mathbf{p}, \omega_i) \cos\theta$$

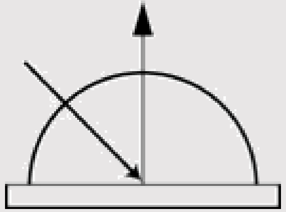
- When people talk about BRDFs, think materials!
  - Graphics is about seeing things
  - How we see a BRDF defines how we see a material

Types of Reflectance Functions

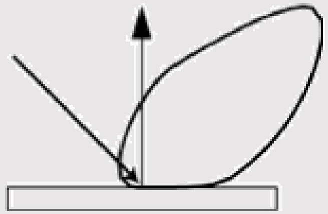
Ideal Specular

- Perfect mirror

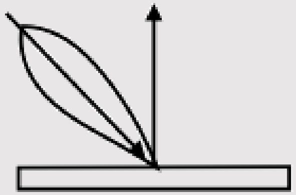
Ideal Diffuse

- Uniform in all directions

Glossy Specular

- Majority of light in reflected direction

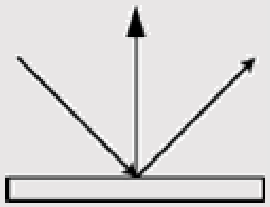
Retroreflective

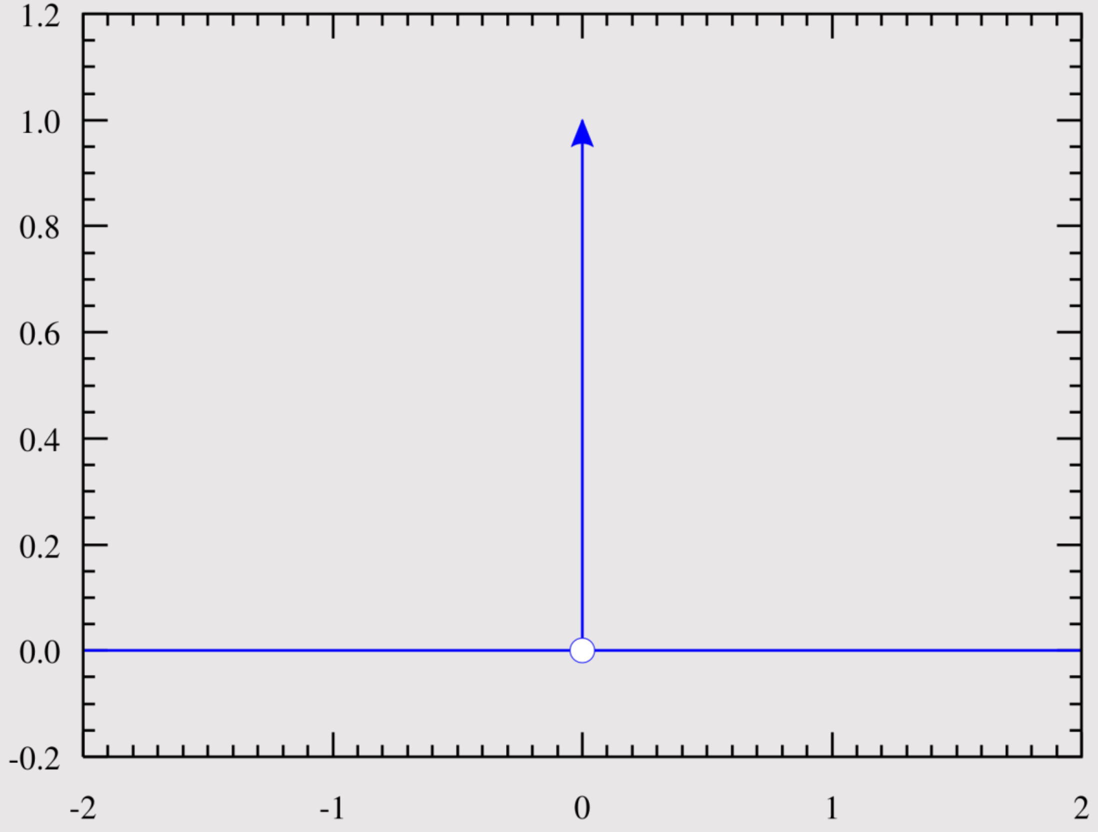
- Reflects light back towards source

Dirac Delta Distribution

- With ideal specular, the BRDF is a constant maximum reflectance (no energy absorbed) in the reflected direction
  - $f_r(\mathbf{p}, \omega_i \rightarrow \omega'_i) = < 1.0, 1.0, 1.0 >$
    - ω'_i is the incoming direction reflected about intersection point **p**'s normal

- Can represent the PDF of an ideal specular as a **dirac delta (δ) function**
  - 1 in one place, 0 everywhere else

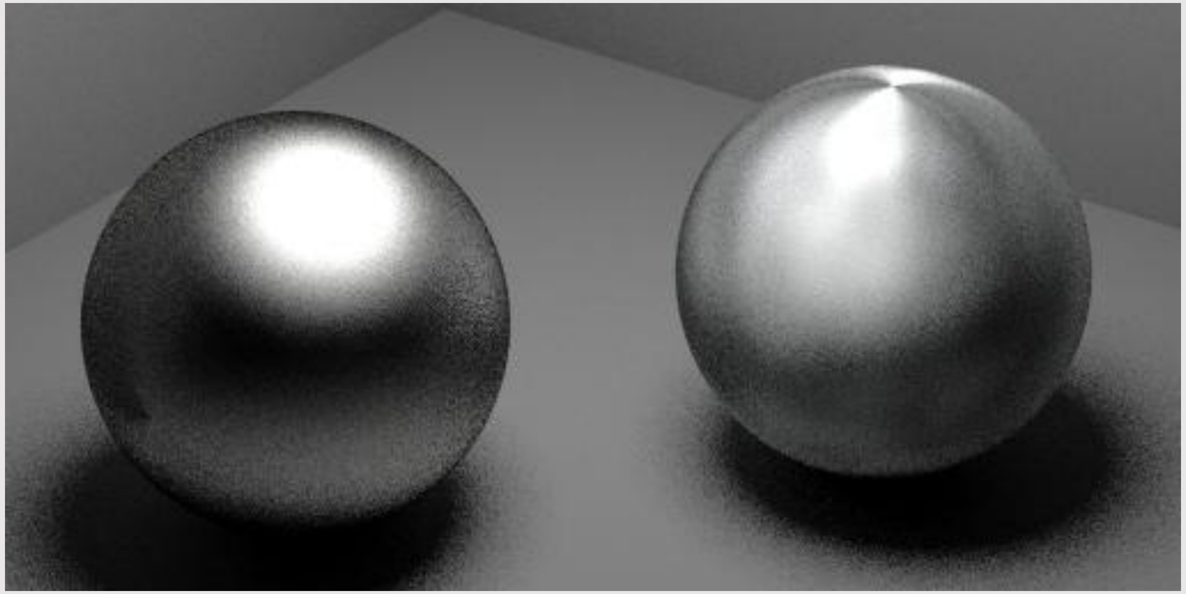




Reflectance Direction

- **Isotropic BRDFs** are fixed when the incident and exiting directions are rotated about the normal

- **Anisotropic BRDFs** vary when the incident and exiting directions are rotated about the normal

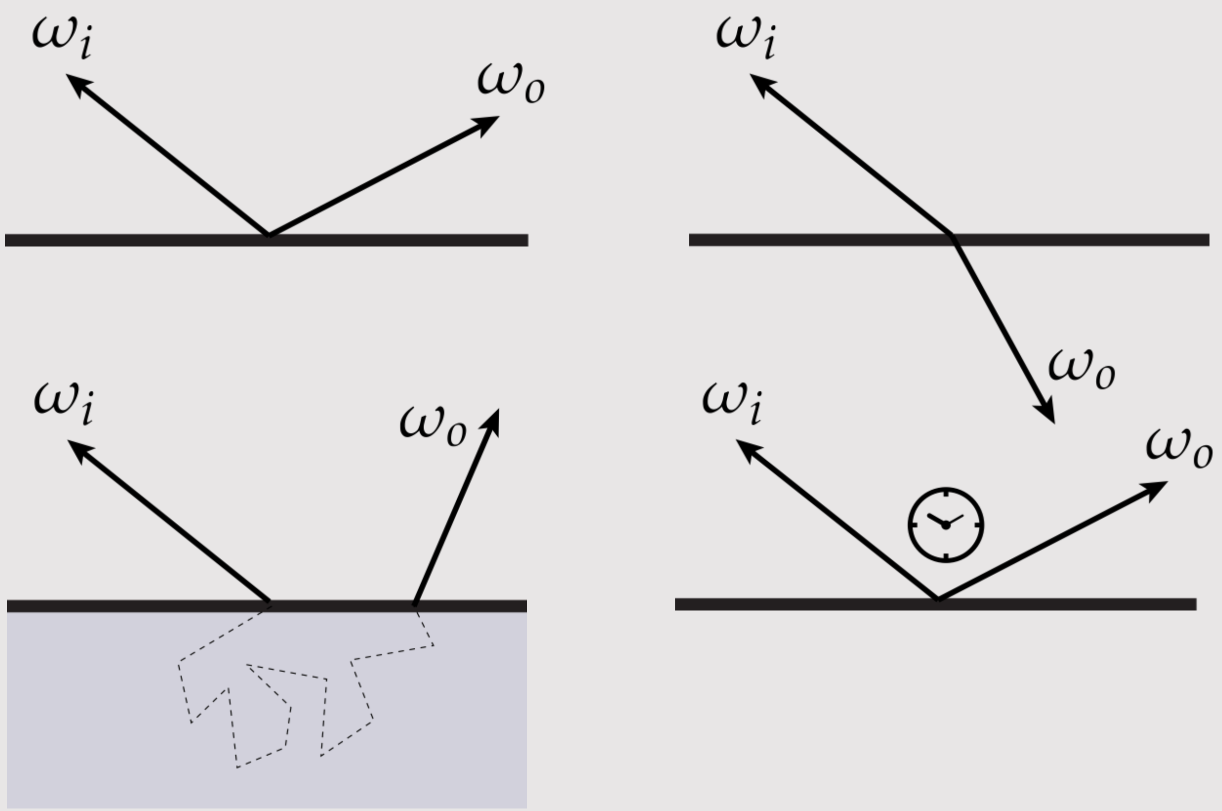


[isotropic] **[anisotropic]**

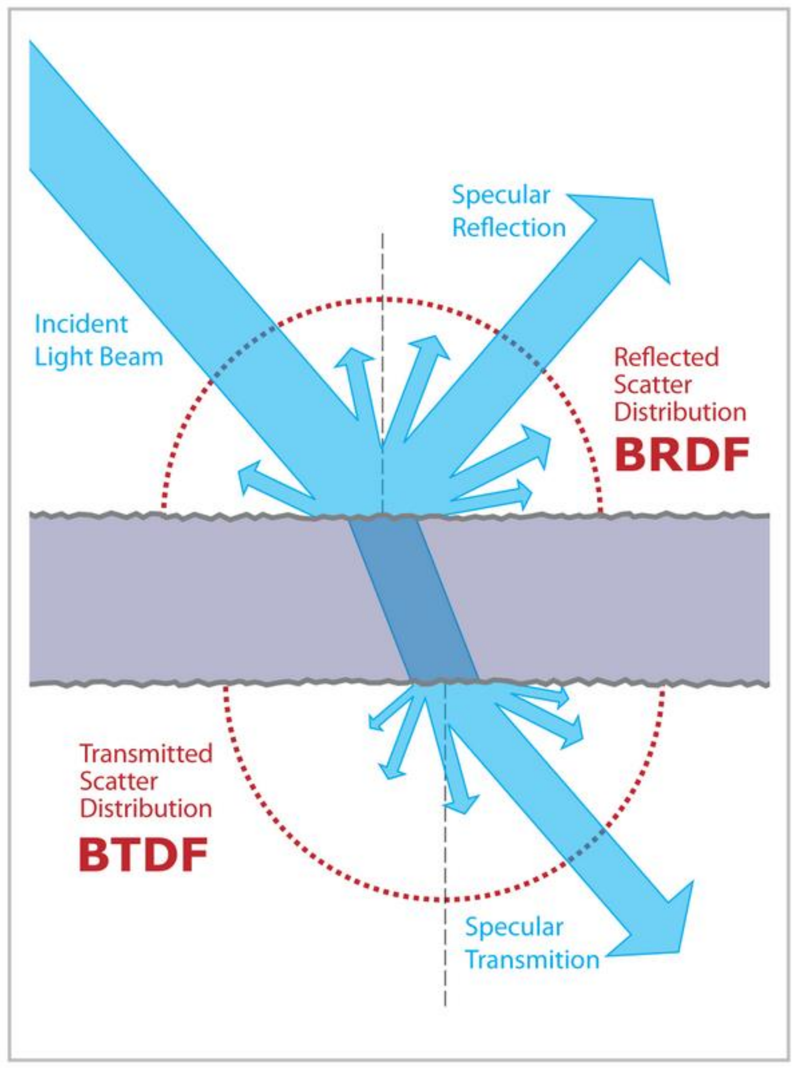
Models Of Scattering

- How can we model "scattering" of light?
- Many different things could happen to a photon:
  - Bounces off surface
  - Transmitted through surface
  - Bounces around inside surface
  - Absorbed and re-emitted

- What goes in must come out!
  - Total energy must be conserved

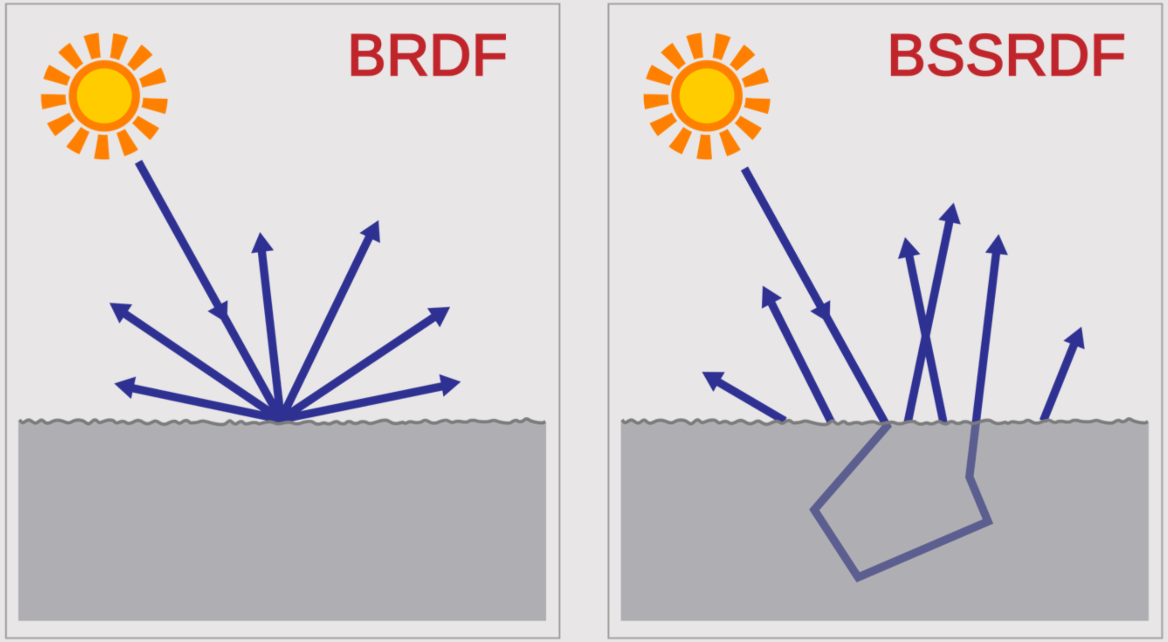


Much More Than Just A BRDF



- **BRDFs** - Bidirectional Reflectance Distribution Function
  - Describes light reflecting without entering the surface
  - Ex: lambertian, mirror
- **BTDFs** - Bidirectional Transmittance Distribution Function
  - Describes light entering the surface
  - Ex: glass
- **BSDFs** - Bidirectional Scattering Distribution Function
  - Encapsulates BRDFs and BTDFs
  - BRDFs are just more common in literature :)

Much Much More Than Just A BRDF

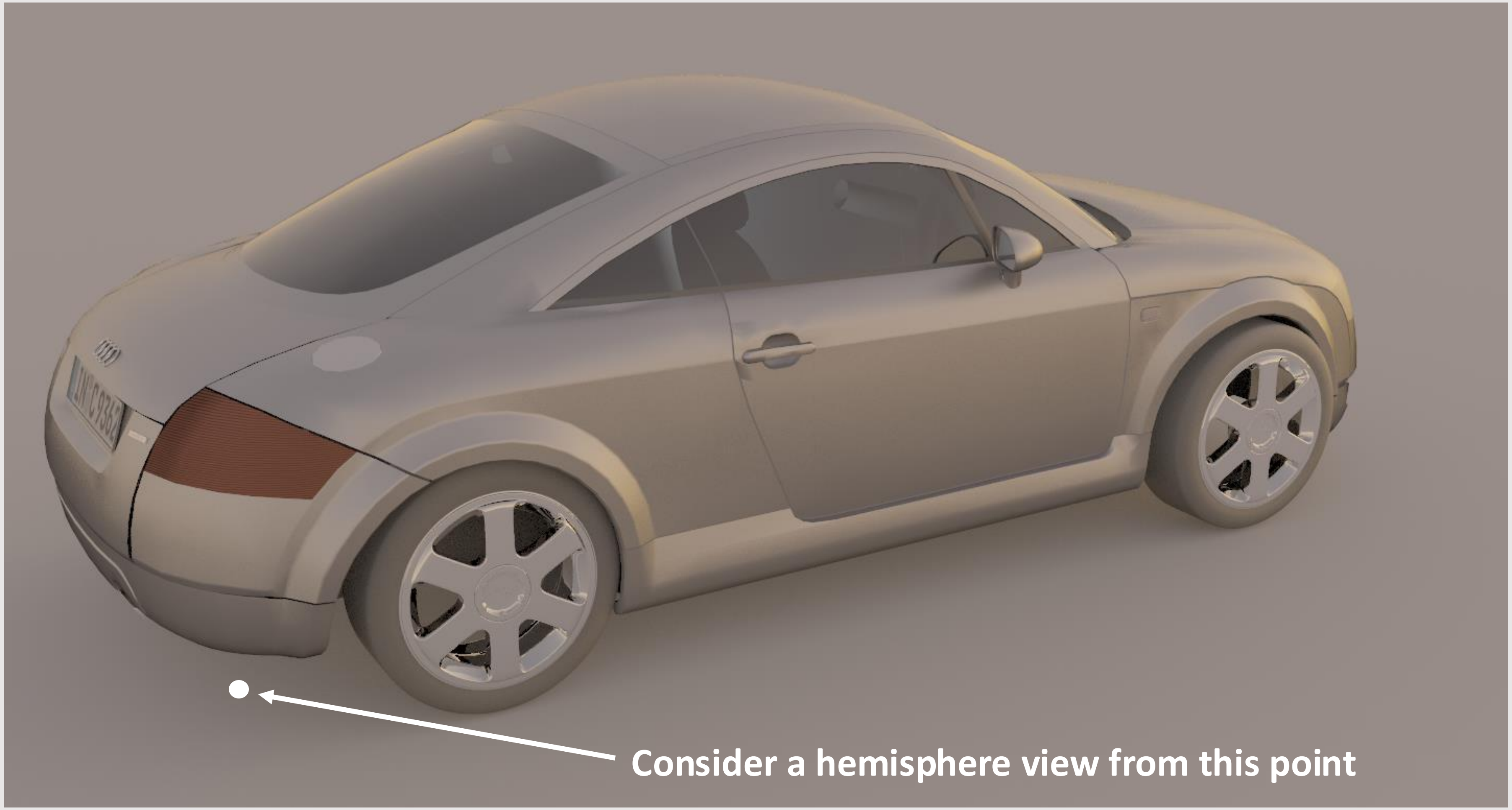


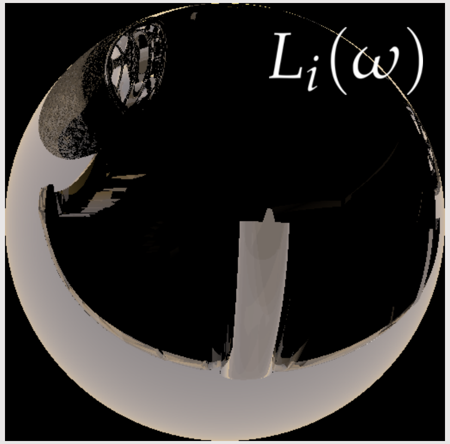
- **BSSRDFs**, *SS - Surface Scattering
    - Describes light entering and scattering the surface before being reflected out
    - Ex: milk
- **BSSTDFs**, *SS - Surface Scattering
    - BTDF but with subsurface scattering
    - Ex: also milk
- **BSSDFs**, *SS - Surface Scattering
    - Encapsulates BSSRDFs and BSSTDFs

BRDF Examples

BRDFs can be a mix of diffuse and specular

Hemispherical Incident Radiance

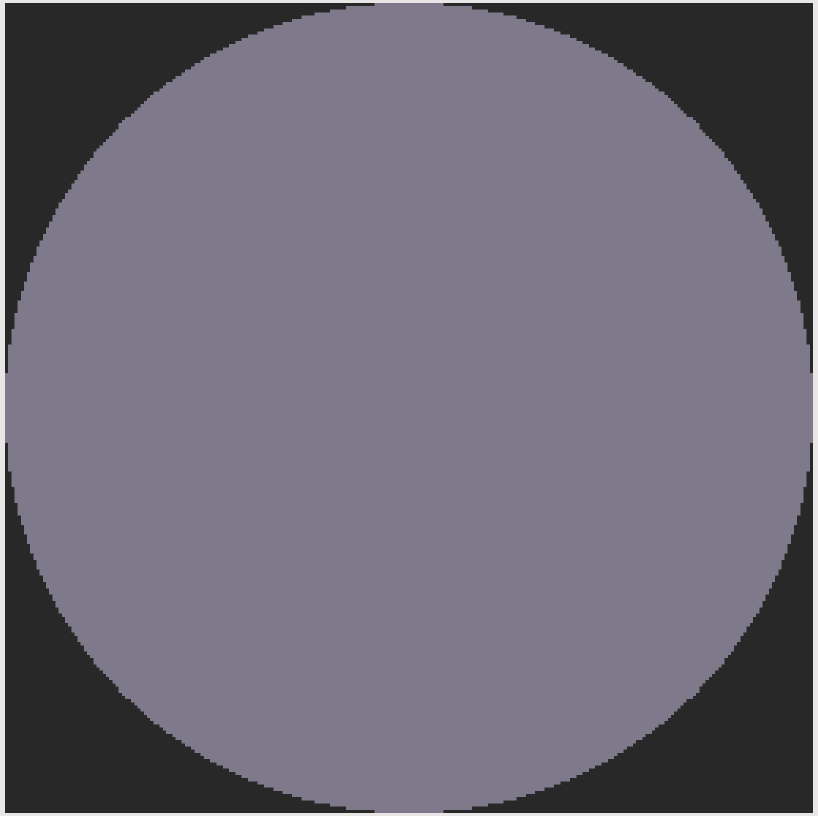




At any point on any surface in the scene, there's an incident radiance field that gives the directional distribution of illumination at the point

Diffuse Exitant Radiance

[incident radiance]

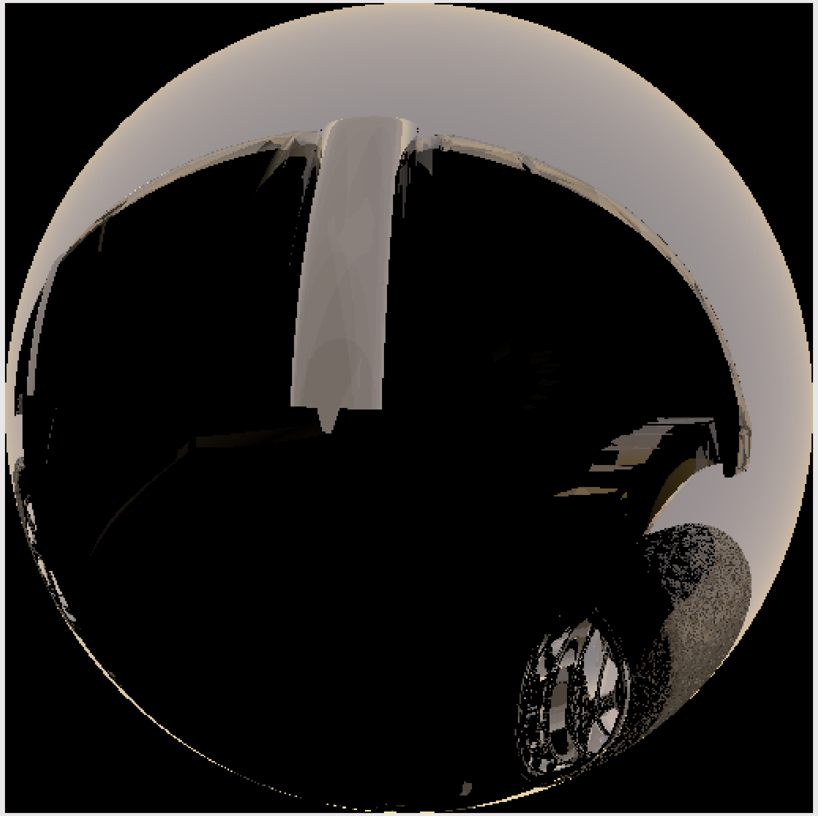


[exitant radiance]

Colors sampled from uniform hemisphere blend all colors into one average color.

Ideal Specular Exitant Radiance

[incident radiance]

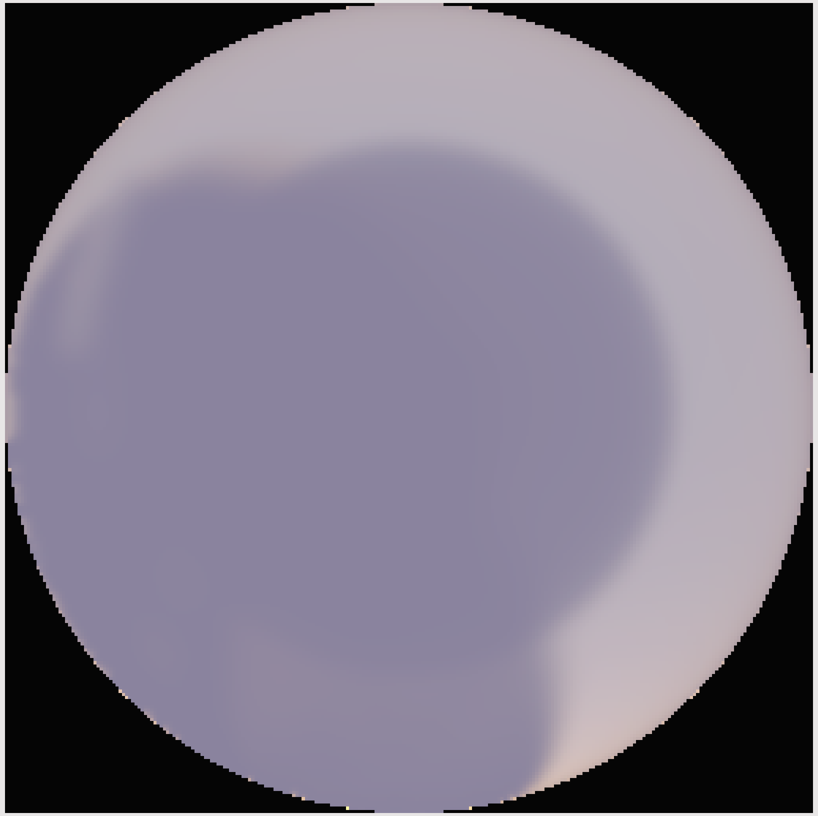


[exitant radiance]

Incident radiance is “flipped around normal” to get exitant radiance.

Plastic Exitant Radiance

[incident radiance]

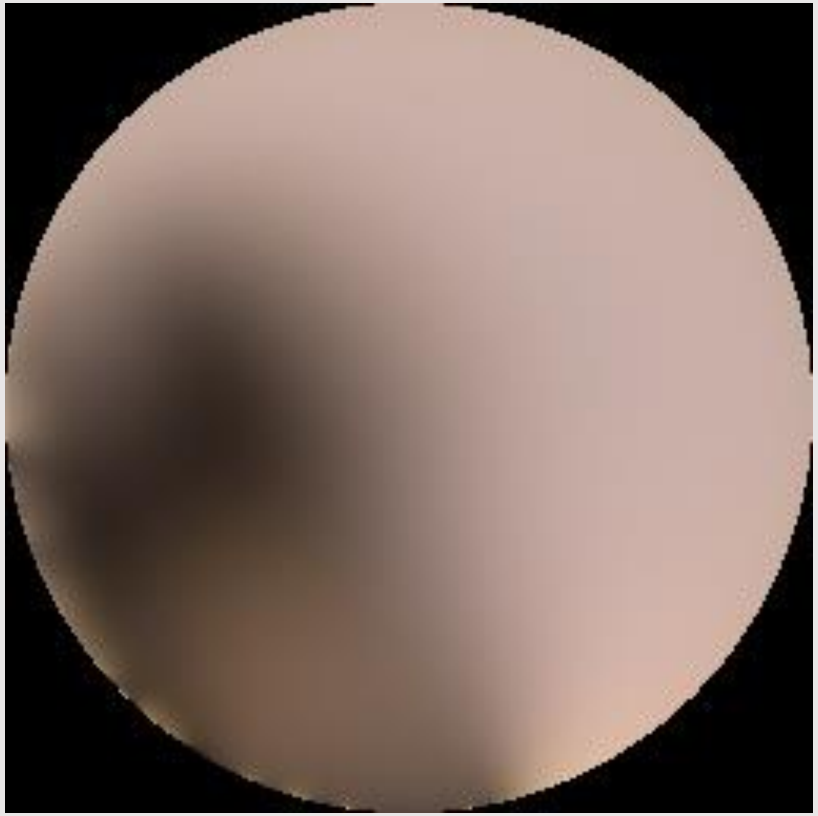


[exitant radiance]

Incident radiance gets flipped and blurred.
Common example of a material that has both diffuse and specular properties.

Copper Exitant Radiance

[incident radiance]



[exitant radiance]

More blurring, plus coloration (nonuniform absorption across frequencies).
Copper absorbs some colors, and emits the rest, giving it a “warm brown” color.

Integration of BRDF

- When integrating the BRDF over the hemisphere, total value will be less than or equal to 1

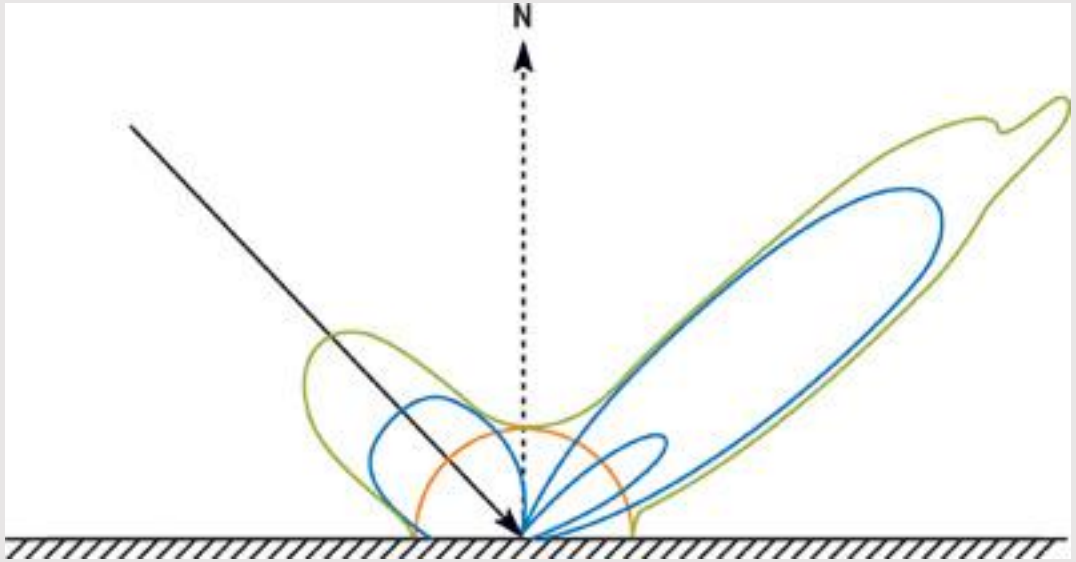
$$\int_{\mathcal{H}^2} f_r(\omega_i \to \omega_o)\ \cos\theta\, d\omega_i \leq 1$$

- Conservation of energy: outgoing energy should be less than or equal to incoming energy
  - Energy should not be created
  - Energy lost is absorbed into the intersected material
    - BRDF helps capture that absorption

- BRDF can never be negative

$$f_r(\omega_i \to \omega_o) \geq 0$$

  - A negative BRDF would imply negative energy???



Radiometric Description of BRDF

- **Recall:** differential irradiance landing on surface from differential cone of directions ω_i

$$dE(\omega_i) = dL(\omega_i) cos\, \theta_i$$

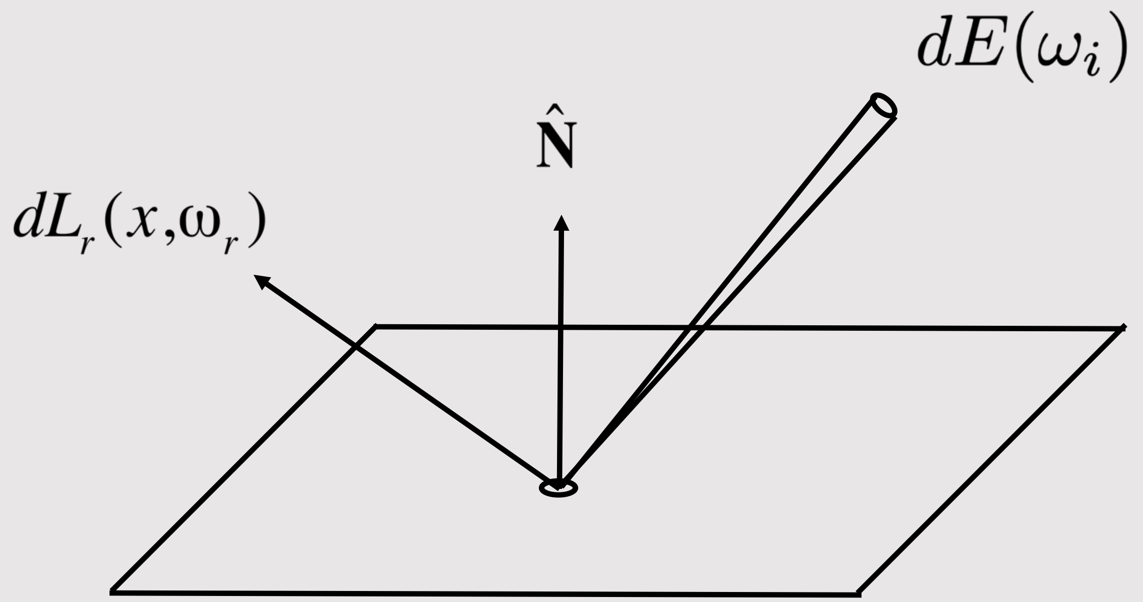
- **Recall:** differential radiance reflected in direction ω_r (due to differential irradiance from ω_i)

$$dL_r(\omega_r)$$

- BRDF captures the ratio between the incoming irradiance and the outgoing radiance

$$f_r(\omega_i \rightarrow \omega_o) = \frac{dL_o(\omega_o)}{dE_i(\omega_i)} = \frac{dL_o(\omega_o)}{dL_i(\omega_i)\cos\theta_i} \left[\frac{1}{sr}\right]$$

  - Given the incoming irradiance, computes the outgoing radiance



$$dL_r(\omega_r) \propto dE_i(\omega_i)$$

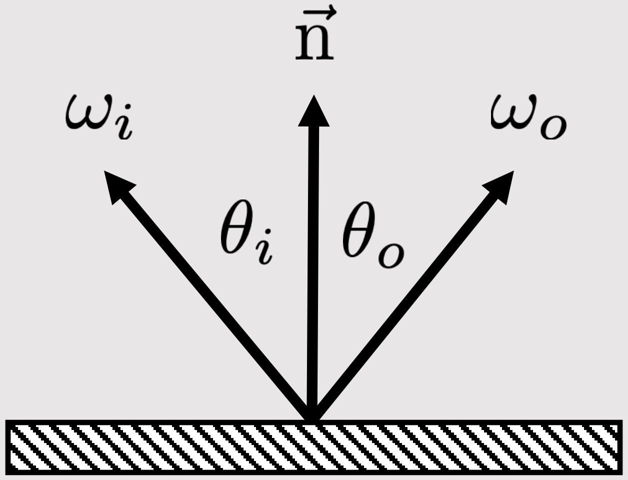
measured in steradians

- ~~BRDFs~~
- Materials
- Environment Lighting

Change Of Syntax

- **Surface-local space**
  - Normal is $n = < 0, 1, 0 >$
  - Unit directions w_i and w_o point away from intersection point p

- All material interactions will occur in surface-local space
  - Transform w_i to surface-local space
  - Compute new outgoing ray w_o
  - Transform w_o back to world space



$$\theta = \theta_o = \theta_i$$

Lambertian Material

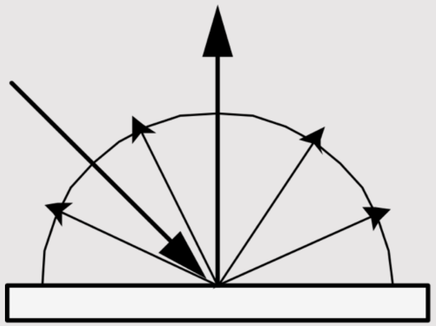
- Also known as diffuse
- Light is equally likely to be reflected in each output direction
  - BRDF is a constant, relying on **albedo** (ρ)

$$f_r = \frac{\rho}{\pi}$$

  - BRDF can be pulled out of the integral

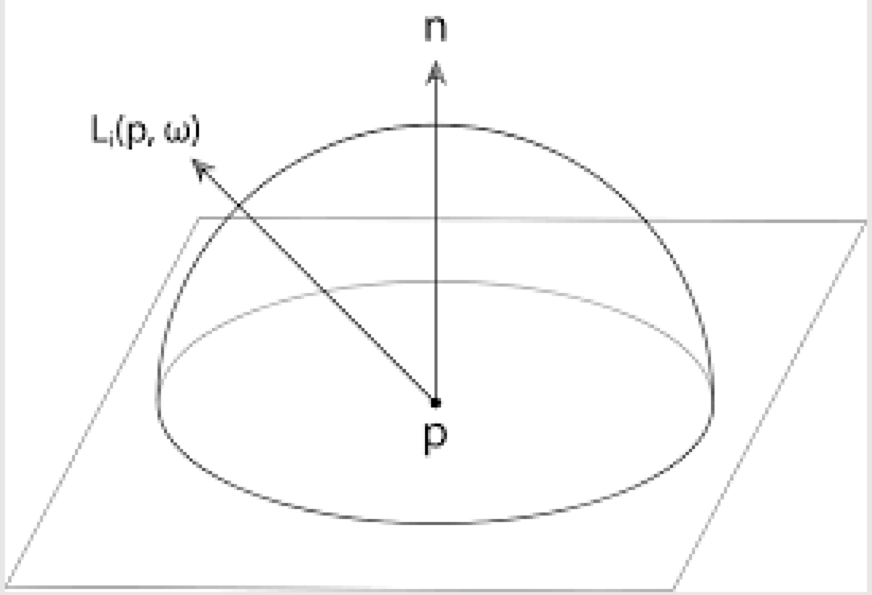
$$\begin{aligned} L_o(\omega_o) &= \int_{H^2} f_r\, L_i(\omega_i)\ \cos\theta_i\, \mathrm{d}\omega_i \\ &= f_r \int_{H^2} L_i(\omega_i)\ \cos\theta_i\, \mathrm{d}\omega_i \\ &= f_r E \end{aligned}$$

- Easy! Pick any outgoing ray w_o



Minions (2015) Illumination Entertainment

Lambertian Material



- The **albedo** (ρ) describes how much of each color is is reflected

- **Why does the Lambertian PDF divide by π?**
  - Consider our irradiance integral:

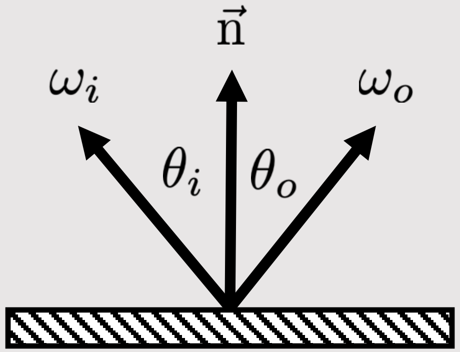
$$\int_{\mathcal{H}^2} f_r(\omega_i \to \omega_o) \ \cos\theta \, d\omega_i \leq 1$$

  - If the albedo is 1, then the integral is greater than 1 (cosine integral over hemisphere is π)
    - Divide the albedo by π to normalize the irradiance so it is less than or equal to 1

$$f_r = \frac{\rho}{\pi}$$

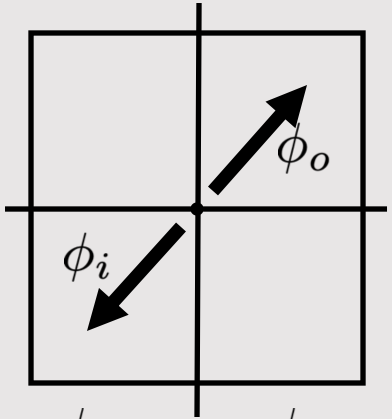
Reflective Material

[side view]



$$\theta = \theta_o = \theta_i$$

[top view]



$$\phi_o = -\phi_i$$

- Reflectance equation described as:

$$\omega_o = -\omega_i + 2(\omega_i \cdot \vec{\mathrm{n}})\vec{\mathrm{n}}$$

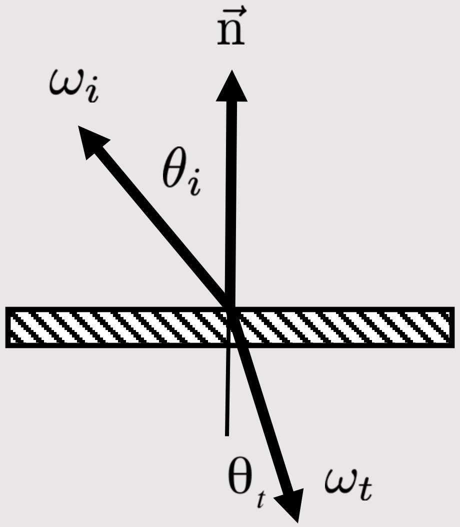
  - Recall incoming and outgoing rays share same origin point **p**

- BRDF represented by dirac delta (δ) function
  - 1 when ray is perfect reflection, 0 everywhere else
  - All radiance gets reflected, nothing absorbed

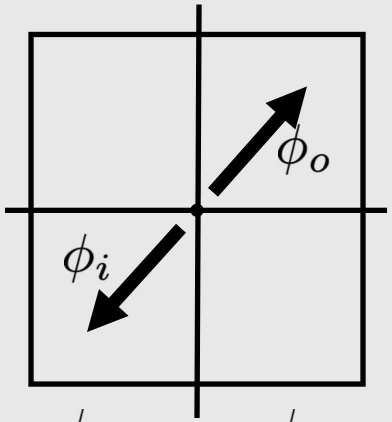
- In practice, no hope of finding reflected direction via random sampling
  - Simply pick the reflected direction!

Refractive Material

[side view]



[top view]



$$\phi_o = -\phi_i$$

- Refractive equation described as:

$$\eta_i \sin\theta_i = \eta_t \sin\theta_t$$

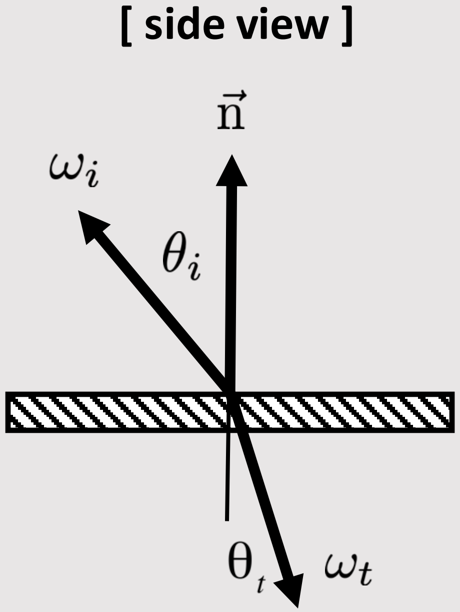
  - Also known as Snell's Law

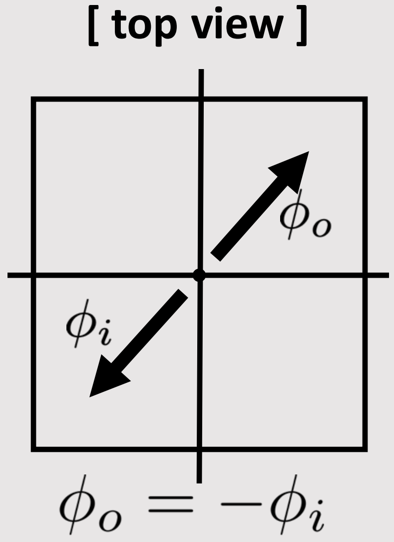
- η_i and η_t describe the index of refraction of the incoming and outgoing mediums
  - Example: η_i is air, η_t is water

Medium	$\boldsymbol{\eta}$
Vacuum	1.0
Air (sea level)	1.00029
Water (20°C)	1.333
Glass	1.5-1.6
Diamond	2.42

  - η is the ratio of the speed of light in a vacuum to that in a second medium of greater density
    - The larger the η, the denser the material

Refractive Material





- Refractive equation described as:

$$\eta_i \sin\theta_i = \eta_t \sin\theta_t$$

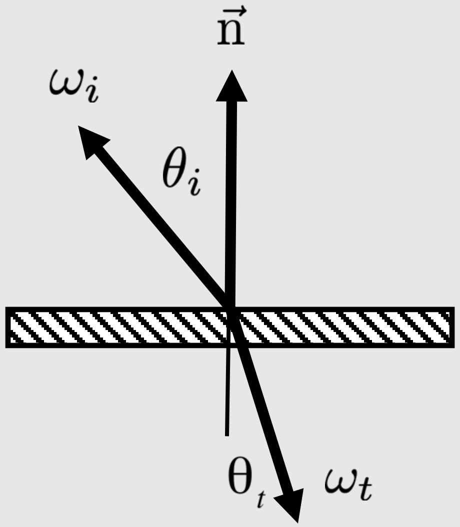
  - Also known as Snell's Law

- Can rewrite the equation as:

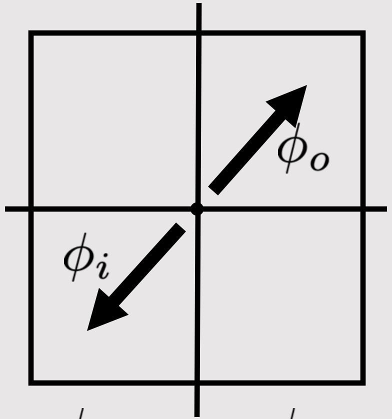
$$\begin{aligned}\cos\theta_t &= \sqrt{1-\sin^2\theta_t} \\ &= \sqrt{1-\left(\frac{\eta_i}{\eta_t}\right)^2 \sin^2\theta_i} \\ &= \sqrt{1-\left(\frac{\eta_i}{\eta_t}\right)^2 (1-\cos^2\ \theta_i)}\end{aligned}$$

Refractive Material

[side view]



[top view]



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- Can rewrite the equation as:

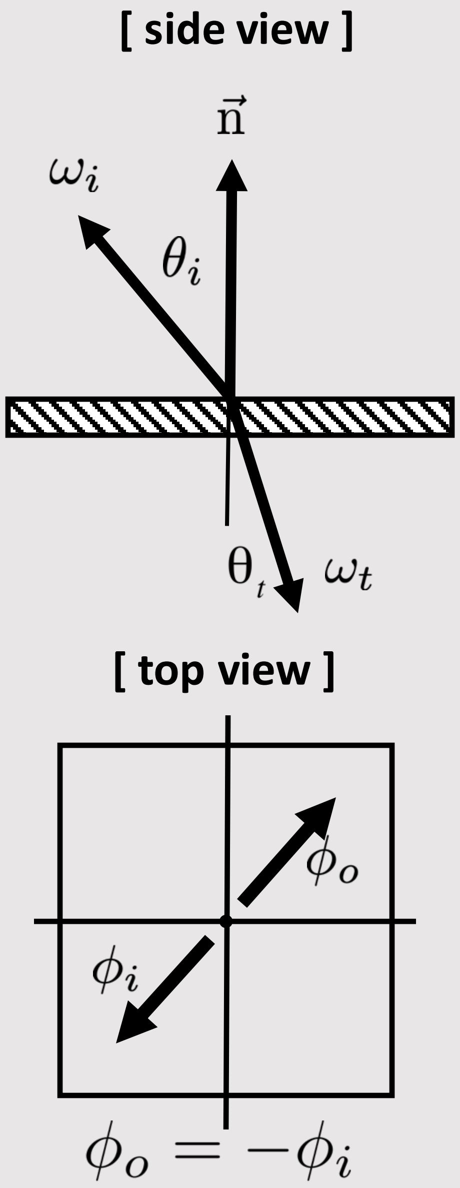
$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t}$$

$$= \sqrt{1 - \left(\frac{\eta_i}{\eta_t}\right)^2 \sin^2\theta_i}$$

$$= \sqrt{1 - \left(\frac{\eta_i}{\eta_t}\right)^2 (1 - \cos^2\theta_i)}$$

what if the term in the square root is negative?

Refractive Material



We know that:

$$0 < cos^2\theta < 1$$

And so:

$$0 < 1 - (1 - cos^2\theta) < 1$$

But if $\eta_i / \eta_t > 1$ then it is possible that:

$$1 - \left(\frac{\eta_i}{\eta_t}\right)^2 (1 - \cos^2\theta_i) < 0$$

This is known as **total internal reflection**, and happens when the incoming index η_i is denser than the outgoing index η_t

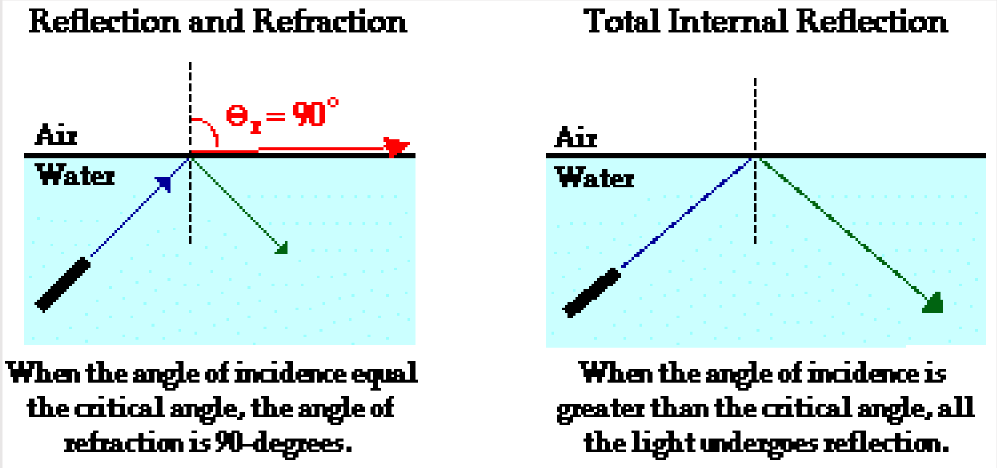
Hence $\eta_i / \eta_t > 1$

Total Internal Reflection

- When going from a more dense (i.e water) to less dense (i.e air) material, light will bend more towards the horizon
  - The incident angle that causes an outgoing 90deg angle is the **critical angle**
  - Can solve for critical angle by solving for θ:

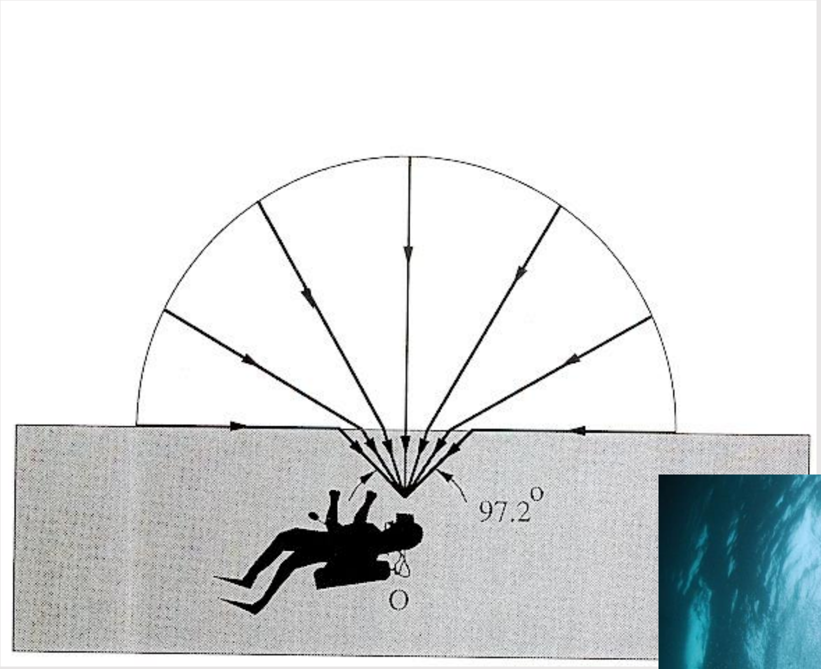
$$1 - \frac{{\eta_i}^2}{{\eta_t}^2}(1 - cos^2\theta) = 0$$

  - When the critical angle is exceeded, the ray is reflected back into the surface



reflections at surface of water viewing from under the surface

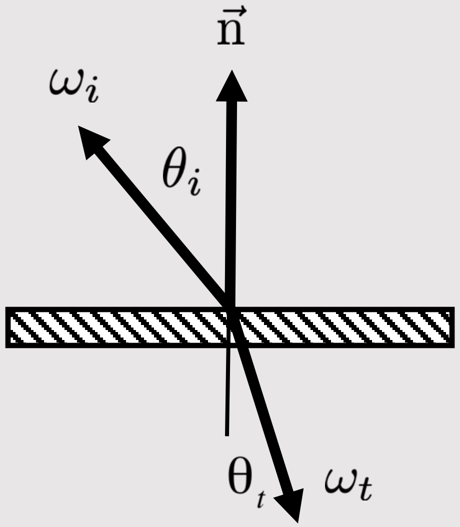
Optical Manhole



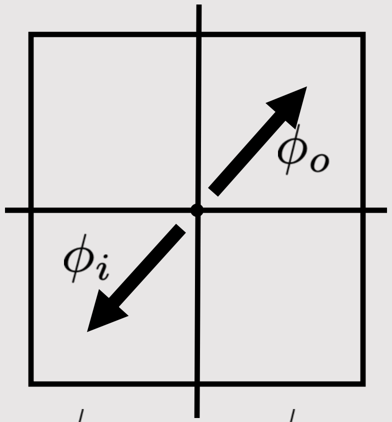
- Works the other direction too
  - Light rays from air entering water bend themselves into a smaller solid angle
    - Pitch black in surrounding areas
  - Gives the illusion that light is a small cone around user

Refractive Material

[side view]



[top view]



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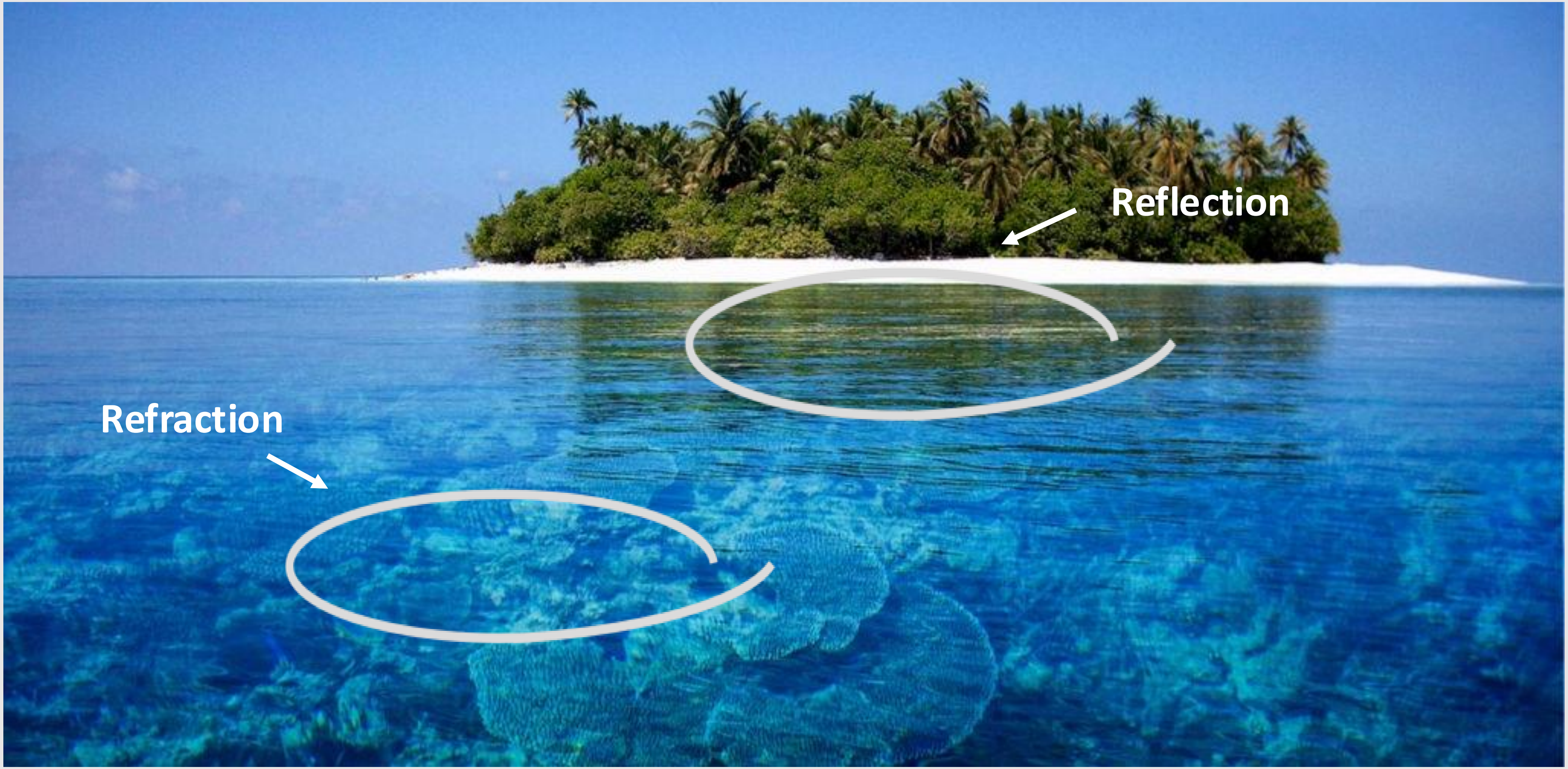
- BRDF represented by dirac delta (δ) function
  - 1 when ray is perfect refraction, 0 everywhere else
  - **Edge Case:** 1 when ray is total internal reflection
  - All radiance gets reflected, nothing absorbed

- In practice, no hope of finding refracted direction via random sampling
  - Simply pick the refracted direction!

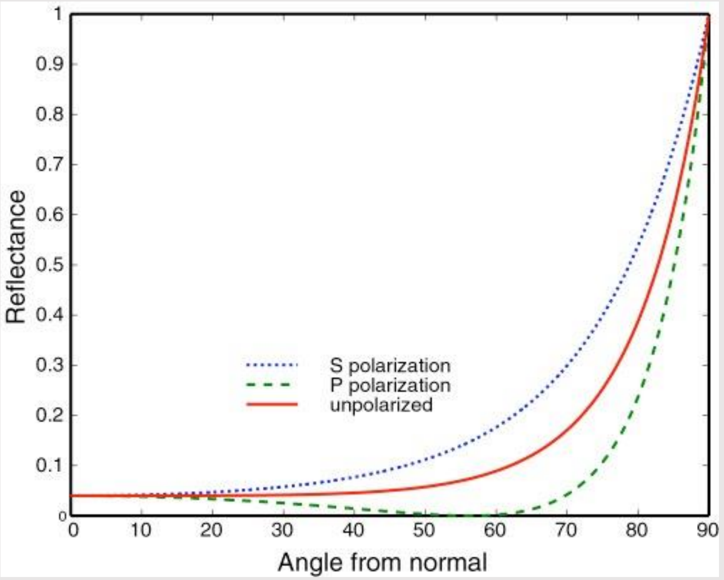
Refractive Isn't Just Refractive



Fresnel Reflectance

- The amount of reflectance increases for refractive material as the angle from the normal increases
  - i.e the angle gets steeper

- Known as the **Fresnel coefficient**



Lafortune et al. (1997)

Fresnel Coefficient

Computing the Fresnel coefficient is kinda hard…

The reflectance for s-polarized light is

$$R_\mathrm{s} = \left|\frac{Z_2\cos\theta_\mathrm{i} - Z_1\cos\theta_\mathrm{t}}{Z_2\cos\theta_\mathrm{i} + Z_1\cos\theta_\mathrm{t}}\right|^2,$$

while the reflectance for p-polarized light is

$$R_\mathrm{p} = \left|\frac{Z_2\cos\theta_\mathrm{t} - Z_1\cos\theta_\mathrm{i}}{Z_2\cos\theta_\mathrm{t} + Z_1\cos\theta_\mathrm{i}}\right|^2,$$

where Z_1 and Z_2 are the wave impedances of media 1 and 2, respectively.

We assume that the media are non-magnetic (i.e., $\mu_1 = \mu_2 = \mu_0$), which is typically a good approximation at optical frequencies (and for transparent media at other frequencies).[3] Then the wave impedances are determined solely by the refractive indices n_1 and n_2:

$$Z_i = \frac{Z_0}{n_i},$$

where Z_0 is the impedance of free space and $i = 1,2$. Making this substitution, we obtain equations using the refractive indices:

$$R_\mathrm{s} = \left|\frac{n_1\cos\theta_\mathrm{i} - n_2\cos\theta_\mathrm{t}}{n_1\cos\theta_\mathrm{i} + n_2\cos\theta_\mathrm{t}}\right|^2 = \left|\frac{n_1\cos\theta_\mathrm{i} - n_2\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_\mathrm{i}\right)^2}}{n_1\cos\theta_\mathrm{i} + n_2\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_\mathrm{i}\right)^2}}\right|^2,$$

$$R_\mathrm{p} = \left|\frac{n_1\cos\theta_\mathrm{t} - n_2\cos\theta_\mathrm{i}}{n_1\cos\theta_\mathrm{t} + n_2\cos\theta_\mathrm{i}}\right|^2 = \left|\frac{n_1\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_\mathrm{i}\right)^2} - n_2\cos\theta_\mathrm{i}}{n_1\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_\mathrm{i}\right)^2} + n_2\cos\theta_\mathrm{i}}\right|^2.$$

Schlick's Approximation

Easier to compute :)

Harder to spell : (

According to Schlick's model, the specular reflection coefficient R can be approximated by:

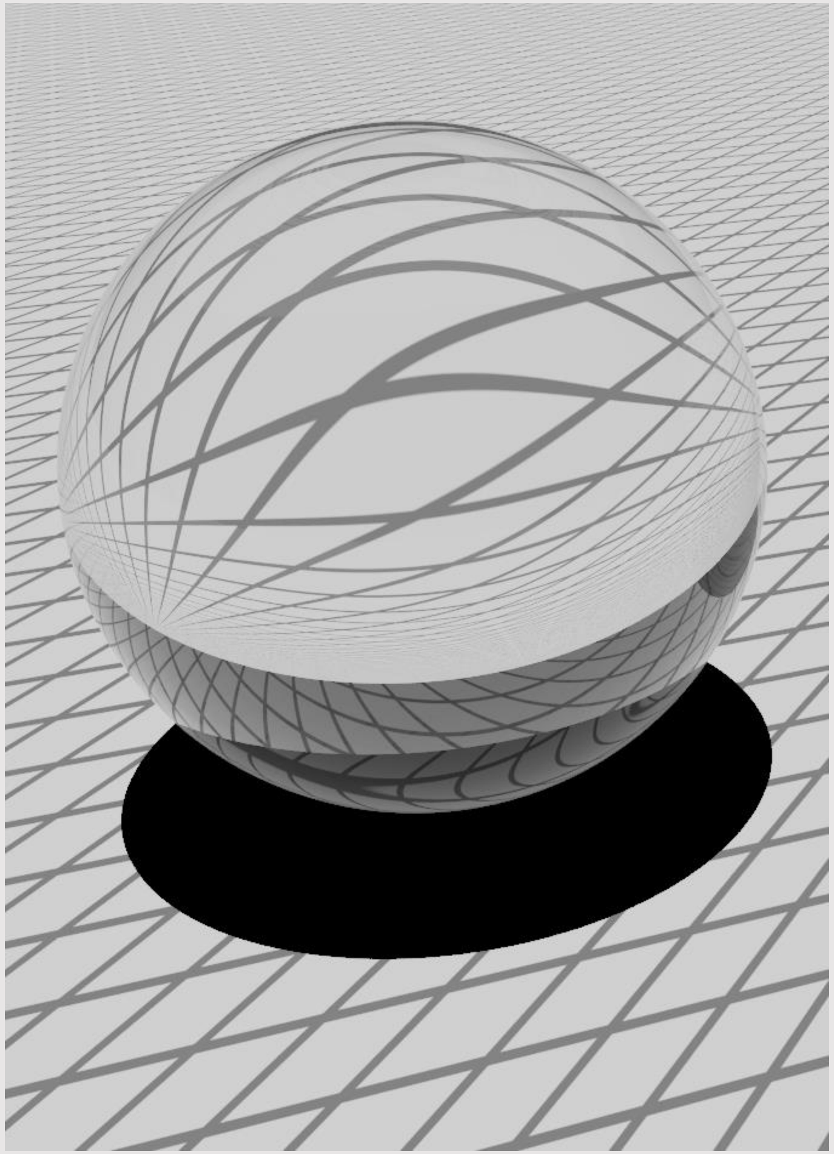
$$R(\theta) = R_0 + (1 - R_0)(1 - \cos\theta)^5$$

where

$$R_0 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

$cos\theta$ is the same as $n \cdot \omega$
for normal n and ray ω

Glass



- Comprised of both reflection (Fresnel) and refraction (Snell)

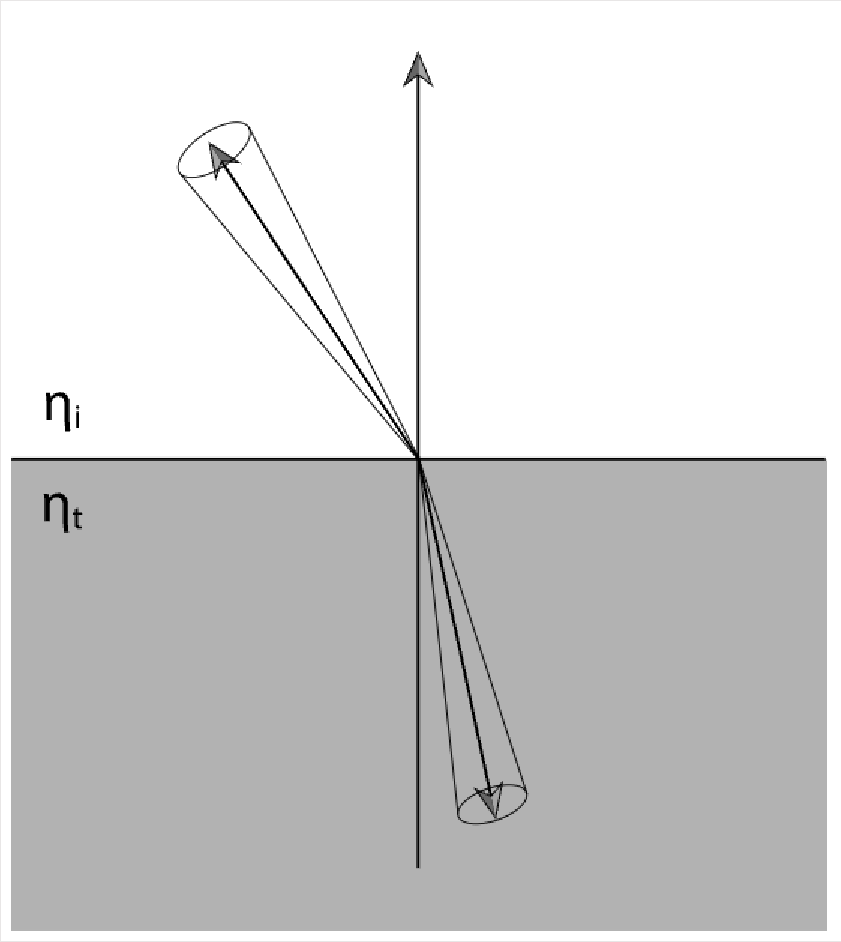
```
void glass(ni, nf, ray ri, ray *rf)
{
  bool internal_reflect = refract(ni, nf, ri, rf);
  if(internal _reflect) {
    // if refraction fails, reflect
    reflect(ri, rf);
    return;
  }

  // compute Fresnel for probability split
  float fr = fresnel(ni, nf, *rf);
  float p = rand();
  if (p < fr) {
    // fr% chance of reflecting
    reflect(ri, rf);
  }
  else {
    // 1 - fr% chance of refracting
    // already refracted, nothing left to do
  }
}
```

Glass

- Transmitted radiance is not the same along boundaries
  - Solid angle is compressed/expanded when changing material IORs

- Need to multiply attenuation by a factor to account for compression/expansion



Glass

Transmitted flux:

Eq 1:

$$d\phi_o = \tau \, d\phi_i$$

Redefine in terms of radiance:

Eq 2:

$$L_0 \, cos\theta_o \, dA \, d\omega_0 = \tau \, (L_i \, cos\theta_i \, dA \, d\omega_i)$$

Convert solid angles to spherical angles:

Eq 3:

$$L_0 \, cos\theta_o \, dA \, sin\theta_o \, d\theta_o \, d\varphi_o = \tau \, (L_i \, cos\theta_i \, dA \, sin\theta_i \, d\theta_i \, d\varphi_i)$$

Differentiate Snell's law w.r.t θ:

Eq 4:

$$\eta_0 \, cos\theta_o \, d\theta_o = \eta_i \, cos\theta_i \, d\theta_i$$

Eq 5:

$$\frac{cos\theta_o \, d\theta_o}{cos\theta_i \, d\theta_i} = \frac{\eta_i}{\eta_0}$$

$\boldsymbol{d\varphi_o = d\varphi_i}$

Substitute above equation and Snell's law into Eq. 3:

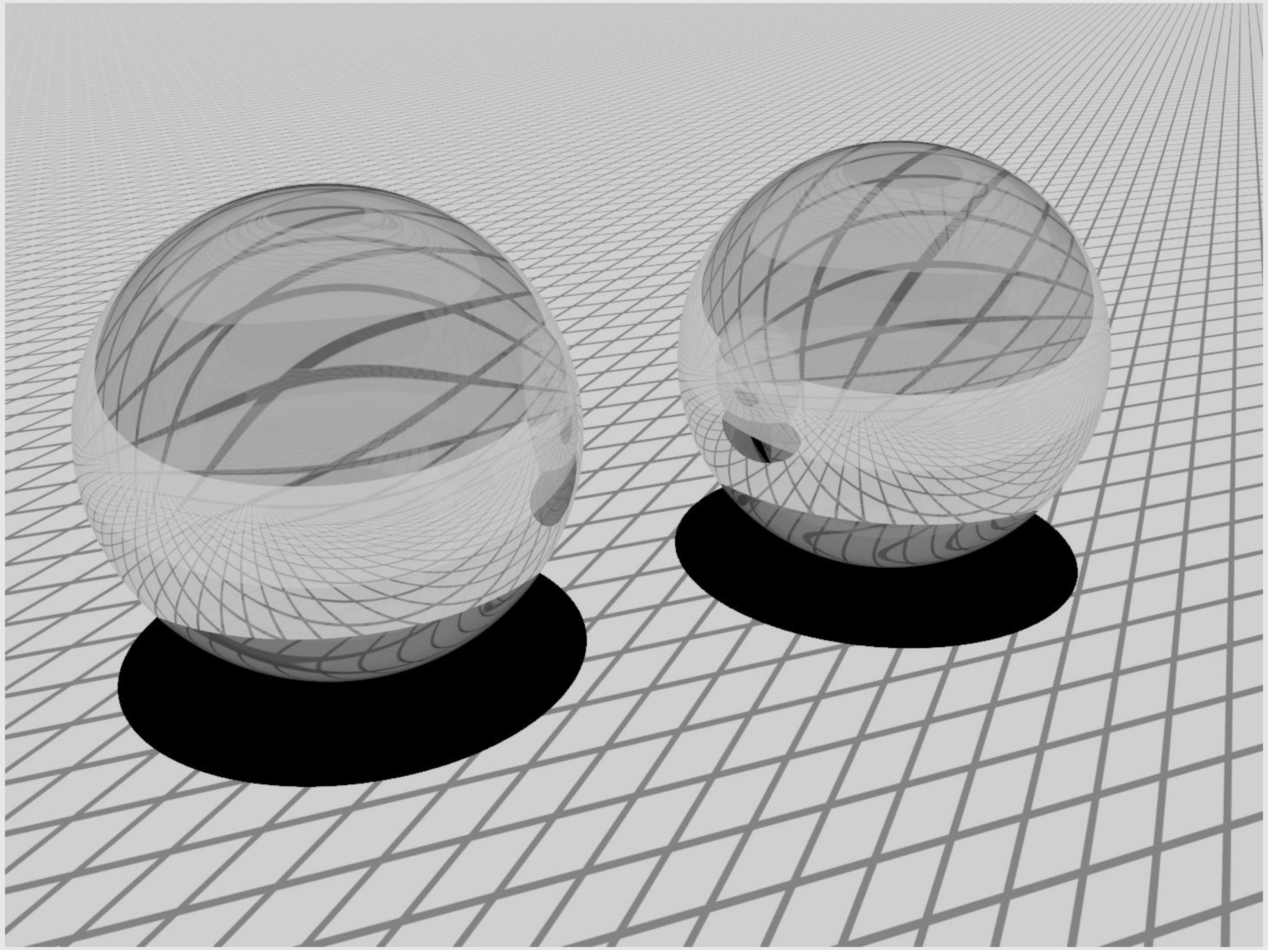
Eq 6:

$$L_0 \, {\eta_i}^2 d\varphi_o = \tau \, (L_i \, {\eta_o}^2 d\varphi_i)$$

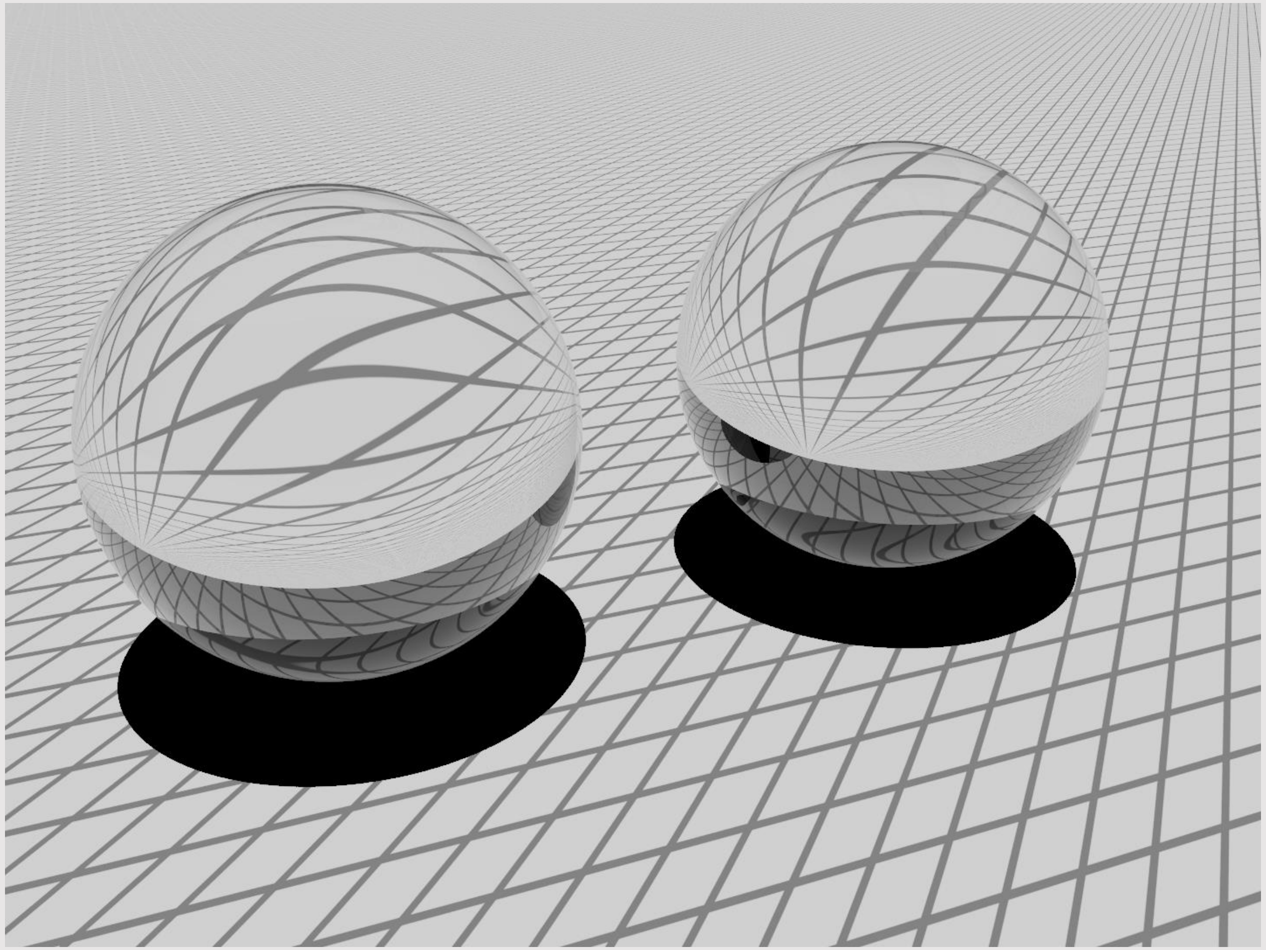
Eq 7:

$$L_0 = \tau \, (L_i \frac{{\eta_i}^2}{{\eta_o}^2})$$

Glass



[without Fresnel]
constant reflection



[with Fresnel]
varying reflection

- ~~BRDFs~~
- ~~Materials~~
- Environment Lighting

Recall: Environment Light

- Sample light directly from an image
- No intensity falloff. Image distance is at infinity
- Very easy to check for visibility
    - Every point in active area

- We'll learn how to build this ~~in a future lecture~~ now

Uncharted 4 (2016) Naughty Dog

Environment As A Light

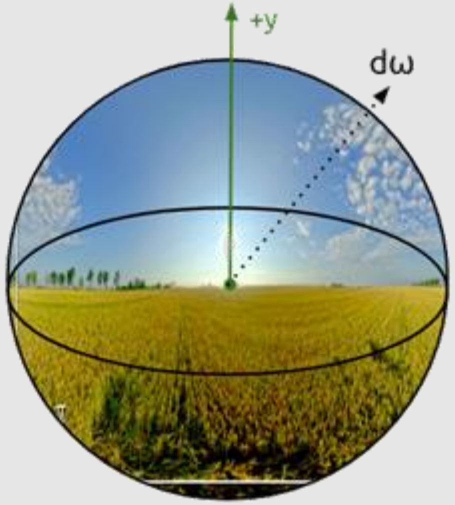

Monster's University (2013) Pixar

- Environment lighting is more than just placing a background image in the scene
  - Scene elements can use the background as a light source, sampling emitted colors the same way we would sample from regular lights
- Saves heavily on compute costs
  - No need to create complex background geometry
  - Think of it as baking diffuse information into a texture and then using that texture as a light
- **Best part:** any image can be used as an environment light!

Polar Coordinates

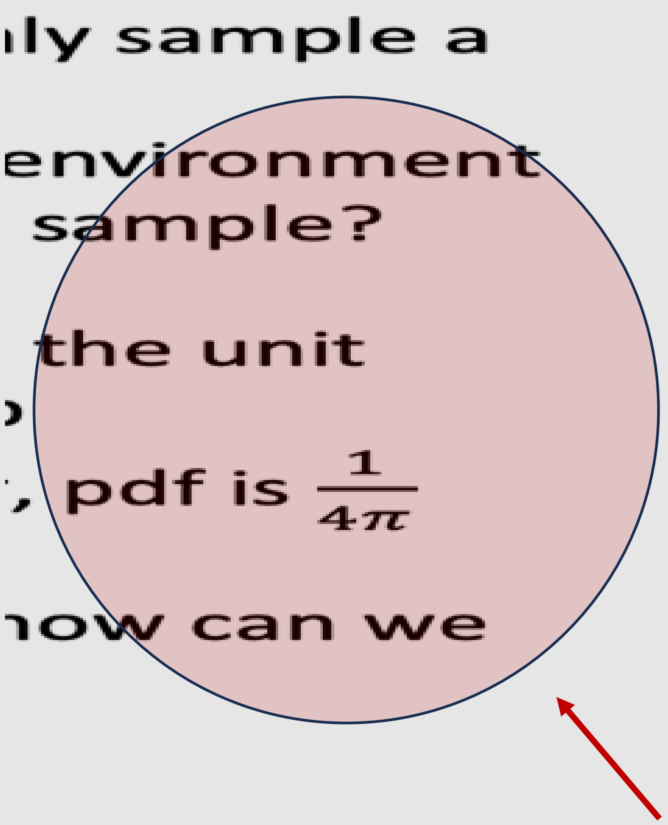
- Normally refer to coordinates on an image by cartesian $[x, y]$ coordinates

- Since we ”wrap” an image around a scene as a sphere, more intuitive to refer to coordinates on an image by polar $[\theta, \varphi]$ coordinates
  - Easy to convert back to cartesian



Uniform Sampling

- When our ray terminates, we randomly sample a light source
  - If the light source we pick is the environment map, where on the image do we sample?
- **Idea:** randomly sample a direction on the unit sphere, trace ray to environment map
  - Surface area of unit sphere is 4π, pdf is $\frac{1}{4\pi}$
- Scotty3D has a hemisphere sampler, how can we extend that to a sphere sampler?
  - Flip a coin, flip the sign
  - Cut the pdf in half: $\frac{1}{2\pi} * \frac{1}{2} = \frac{1}{4\pi}$



focus on all areas equally

Uniform Sampling

```
void env_lighting(ray ri)
{
  // generate ray uniformly
  ray rf = hemisphere::sampler();
  // half chance of flipping ray
  // our "clever" sphere sampler
  float p = rand();
  if (p > 0.5) {
    rf.y = -rf.y;
  }

  // double the options, half the pdf
  float pdf = hemisphere::pdf() * 0.5;

  // trace ray into environment map
  trace_ray(rf);
}
```

- Why do we need to trace the environment lighting ray? Can just sample image pixel
  - Environment lighting ray may still be occluded by scene geometry!

Uniform Sampling

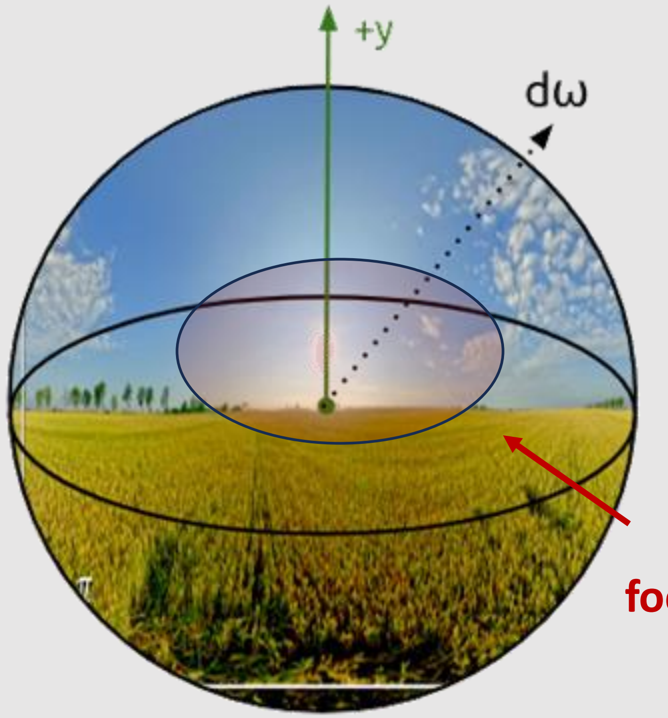
- **Issue:** uniform sampling takes a long time to converge
  - Mixing dark regions of the image with light regions
    - Gives appearance of high variance
  - Will converge with enough samples, but needs a lot of samples

- Is there another approach we can use that prioritizes areas with high info (light)?

Importance Sampling

- **Idea:** sample a direction on the unit sphere proportional to the luminance at that pixel
  - Brighter areas are more important
- **Algorithm:**
  - Assign a probability to each pixel proportional to its luminance
  - Use inversion sampling to pick a sample based on the new probability distribution
  - Create and trace a ray to pixel
  - Divide contribution by PDF of selected pixel
    - Division helps keep sampler unbiased

Will learn more about this next lecture



focus on bright regions

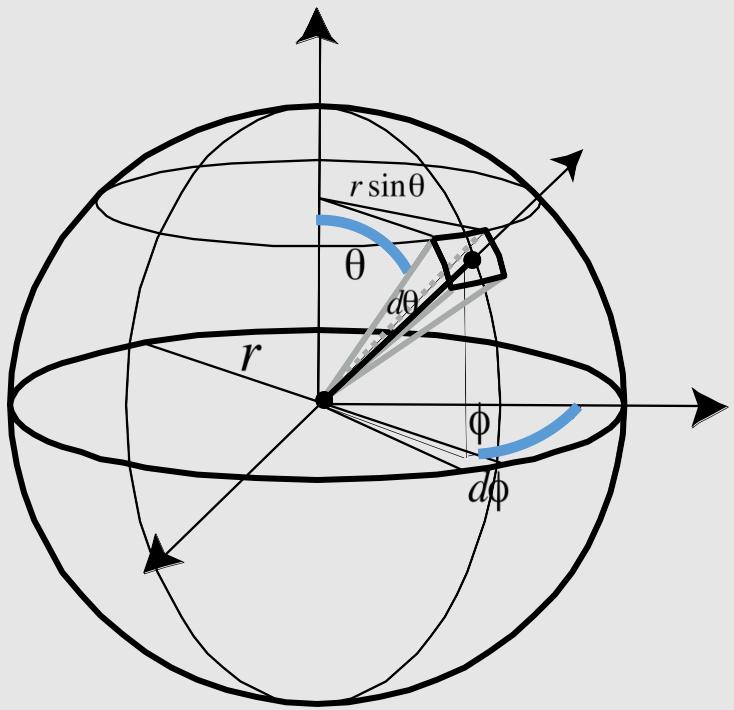
Creating A PDF

- PDF of a pixel should be proportional to its flux
  - Flux = luminance $*$ solid angle
  - Luminance is L
  - Solid angle is $sin\theta d\theta d\varphi$
    - Area for each pixel is the same
    - Simplifies to $Lsin\theta$

- Already have a mapping from $[x, y]$ to $[\theta, \varphi]$

- To make sure distribution is valid, need to normalize PDFs
  - Divide every PDF by the sum of all PDFs

- How can we use that info to sample pixel with a discrete probability distribution?



Inversion Sampling

- Convert PDF to CDF:

$$cdf(i) = pdf(i) + cdf(i-1)$$

  - Image is 2D, CDF is 1D
  - Flatten image into 1D array
    - Recall images are 1D in memory

- Generate random number r between 0 and 1
  - Find index i such that:

$$cdf(i-1) < r < cdf(i)$$

    - Convert i to polar coordinates $[\theta, \varphi]$
    - Construct and trace ray from polar coordinates

Importance Sampling

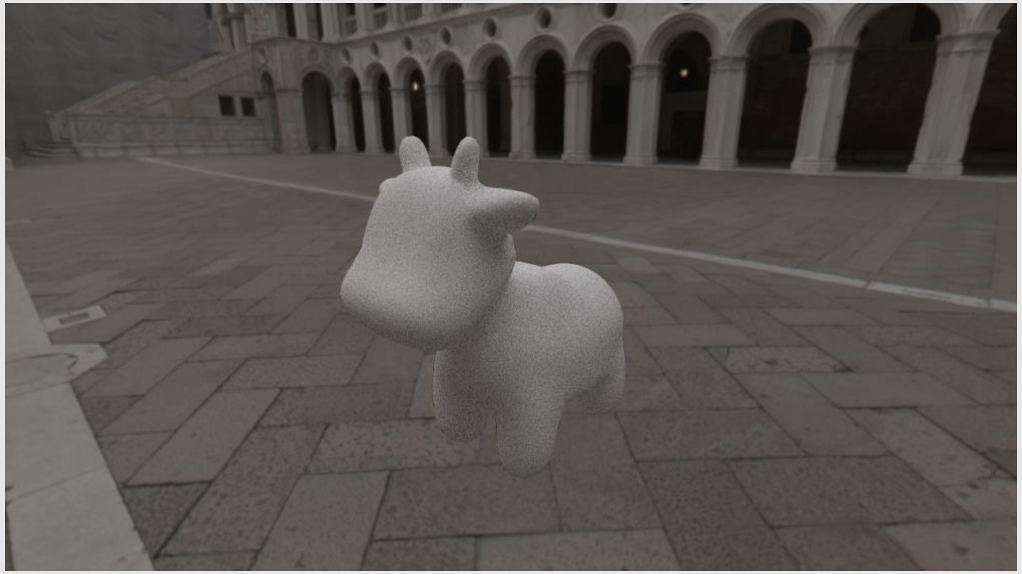
```
void env_lighting(ray ri)
{
  // generate pdf and cdf
  vector<float> pdf = Image::pdf();
  vector<float> cdf = Image::cdf(pdf);

  // inversion sampling
  float p = rand();
  auto i = upper_bound(cdf.begin(),
                       cdf.end(), p);

  // create ray from target pixel
  ray rf = ray_from_index(i);

  // trace ray into environment map
  trace_ray(rf);
}
```

- Notice how we do not even use the incoming ray
  - Both uniform and importance ignore incident directions



Uniform vs. Importance

32 Uniform samples

32 Importance samples

Importance sampling is better able to capture directional light

Uniform vs. Importance

32 Uniform samples

32 Importance samples

Importance sampling is better able to capture sparse lights